TRIGONOMETRY.

AN ELEMENTARY TREATISE ON TRIGONOMETRY,

WITH ITS DIFFERENT APPLICATIONS.

BY

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ADVERTISEMENT.

AS the following Treatise is intended for the use of Students in the University, their interest and convenience have been exclusively considered in drawing it up. It consists of two parts:—the object of the first is the detail of the principles of a general Theory of sines, and the application of these principles to the resolution of plane and spherical triangles; that of the second, the continuation of this Theory, and its applications to the higher departments of Analytic Science. Both parts are handled merely in a Theoretical point of view; numerical practice we have almost totally excluded; it can be had in abundance from the instructions prefixed to the Trigonometrical Tables in common use, and unless the reader be fur-

nished with such tables, any set of numerical examples would be useless.

The Treatise of Professor Woodhouse, which has heretofore served as a text book to the Students of our University, though excellent as a practical system, is very badly calculated to teach the principles of the science; the method is indirect and unnatural, chiefly in consequence of his deducing the Theory of sines from the triangle, which ought to have been deduced from the circle; on the higher applications, his work is exceedingly limited and incomplete: to remedy these inconveniencies, and to produce a Treatise more adapted to the Examinations of the Dublin University, have been the Author's earnest endeavours: how far he has succeeded, he now leaves it to the reader to decide, requesting, at the same time, his indulgence for the imperfections necessarily attendant on a first, and much interrupted publication.

TRINITY COLLEGE,
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PART I.

CHAP. I.

DEFINITIONS AND PRINCIPLES.—ON THE DIVISIONS OF THE CIRCLE, &c.

ART. 1. Let it be assumed that any arc of a circle is less than the sum of the tangents to its extreme points, and greater than its chord. It follows from this, that the perimeter of the circumscribed polygon is always greater, and that of the inscribed polygon less than that of the circle, however the number of sides be multiplied.

2. If a right line be drawn cutting the sides of a triangle in any way above the base and below the vertical angle, the sum of the sides is greater than the sum of the segments towards the base together with the drawn line.

3. Hence by drawing tangents or chords to the middle points of the intercepted arcs, the number of the sides of the inscribed or circumscribed polygons is doubled, but their perimeters are made to approach nearer to that of the circle. If this bisection be continued *indefinitely* the perimeters of the polygons can be made to differ from each other, and $\therefore$ *a fortiori* from that of the circle, by a quantity less than any assignable.

4. The inscribed or circumscribed regular polygons of two circles have their perimeters to each other (*componendo*) as the radii of the circles. Hence we can prove that the circumferences of circles are to each other as their radii. For let C, C', be the circumferences, and r, r', the radii; then if possible let $r : r' :: C : C' + \partial$; let P, P', be the perimeters of the circumscribed polygons at the

time that P' is nearer to C' than by the quantity δ, then $C:C'+\delta::P:P'$, but P' is less than $C'+\delta$ $\therefore$ P is less than C, which is absurd. Again let, if possible, $r:r'::C:C'-\delta$, and let p, p' be the perimeters of the inscribed polygons at the time that p' is nearer to C' than by δ, then $C:C'-\delta::p:p'$, but p' is greater than $C'-\delta$ $\therefore$ p is greater than C, which is absurd. Hence $r:r'::C:C'$.

If $r'=1$, it is agreed to denote C' by the character 2π. By the above proportion, we obtain thus $C=2\pi r$, or the perimeter of a circle whose radius is r in terms of that of the circle whose radius is unity. The numerical value of π is not capable of being computed in finite terms; and indeed is not easily to be computed even by approximation. As a sufficiently near value let us take $\pi=3{,}14159$.

5. Angles A, A' in different circles, are to each other directly as the arcs a, a' that subtend them, and inversely as the radii of the circles. For

$$\left\{\begin{matrix} A:4\,r.\angle s.::a:C \\ 4\,r\angle s.:A'::C':a' \end{matrix}\right\} \therefore A:A'::\left\{\begin{matrix} a:a' \\ C':C \end{matrix}\right\}::\left\{\begin{matrix} a:a' \\ r':r \end{matrix}\right\}$$

6. In the same circle $A:A'::a:a'$, the angles are therefore measured by the arcs that subtend them. In Trigonometry, however, the computation of angles is carried on by means of certain right lines drawn in and about the circle, which lines are not direct measures of the angles as the arcs are, but from which the angles can be deduced.

A *Theoretical* connection between such lines and their respective arcs or angles, the reader will find in the sequel; a *practical* connection is afforded the computist by registered tables, in which the respective values of each are to be found in terms of the other.

7. These lines are denominated *sine*, *tangent*, *secant*, &c.; which we shall now proceed to describe.

The *sine*, is the perpendicular from one extremity of an arc on the radius passing through the other extremity.

The *tangent*, is the portion of the touching line through one extremity of an arc, intercepted by the produced radius through the other extremity.

The produced radius to meet the touching line is called the *secant*.

The *chord* of an arc, is the line joining its extremities.

The *complement* of an arc, is the difference between the arc and a quadrant. The *supplement*, the difference between the arc and two quadrants.

The *sine* of the complement, or distance between the centre and foot of the sine, is called the *cosine*.

The *versed-sine* is the difference of radius and cosine.

The *cotangent* is the tangent of the complement.

The *cosecant* is the secant of the complement.

In (Fig. 1.) taking a circle with a radius $CA = R$, and drawing through C two lines; one AD from right to left, and the other BE from top to bottom at right angles to it; it is agreed to count all arcs from A around the circle in the direction $ABDE$.

Assuming then any arc Am, which we shall denote by the character A; letting fall the perpendicular Mn, drawing a tangent at A and producing the radius to meet the tangent at v, we have, (using the language of Trigonometry),

$Mn = \sin A$	$nC = \cos A$
$Av = \tan A$	$Bv' = \cot A$
$nA = \text{vers-sin } A$	$Bo = \text{co-vers-sin } A$
$Cv = \sec A$	$Cv' = \text{co-sec } A$

The line nD is sometimes denoted by the character Su-versin A.

8. Several simple relations between those lines may be had obviously from the Fig. viz. such as the following:

1° $R^2 = \sin^2 A + \cos^2 A$.

2° By the sim. $\triangle$s vAC, CBv', $\tan A : R :: R : \cot A$, $\therefore$ $\tan A = \frac{R^2}{\cot A}$.

3° $vA : R :: \sin A : \cos A$, $\therefore$ $\tan A = R . \frac{\sin A}{\cos A}$. and also $\cot A = R . \frac{\cos A}{\sin A}$.

4° $\sec A : R :: R : \cos A$, $\therefore \sec A = \frac{R^2}{\cos A}$, and also $\operatorname{cosec} A = \frac{R^2}{\sin A}$.

5° $\tan A : \sec A :: R : \operatorname{cosec} A$, $\therefore \tan A = R . \frac{\sec A}{\operatorname{cosec} A}$, and also $\cot A = R \frac{\operatorname{cosec} A}{\sec A}$.

9. Any lines similarly situated in circles of different radii are to each other as the radii. If the radius of one circle be unity and of the other R; then any line, $\sin A$ for instance, in the latter circle $=$ the similar line in the former multiplied by R.

We may therefore indifferently use in a formula $\sin A$ to radius R, or $R . \sin A$ to radius unity, since they are equal. Or we may indifferently use $\sin A$ to radius unity, or $\frac{\sin A}{R}$ to radius R.

Suppose we have then any formula computed to radius unity, and wish to see its form for radius R, we substitute for each Trigonometrical line the same divided by R; *e, g,* $\cos 3 A = 4 \cos {}^3 A - 3 \cos A$ becomes $\frac{\cos 3 A}{R} = \frac{4 \cos {}^3 A}{R^3} - \frac{3 \cos A}{R}$ or clearing of fractions $R^2 \cos 3 A = 4 \cos {}^3 A - 3 R^2 \cos A$. Hence the following rule.

Multiply each term but the highest by that power of R, *that will make all the terms of the same dimensions.*

10. Agreeing to consider the several Trigonometrical lines positive in the first quadrant; and adopting as a criterion of positive and negative that quantities change their sign in passing through infinity or cypher, we may perceive the following variations of sign.

1° All arcs beginning at A and terminating at any point bebetween A and D have *positive* sines; and terminating between D and A *negative* sines.

2° All arcs that terminate between A and B have *positive* cosines; between B and D *negative*; between D and E *negative*; and between E and A *positive.*

3° All arcs that terminate between A and B have *positive* tangents; all between B and D *negative*; between D and E *positive*; and between E and A *negative*.

4° All arcs that terminate between A and B or E and A have *positive* secants; all between B and D or D and E *negative*.

5° All arcs that terminate between A and D have *positive* cosecants, all between D and A *negative*.

'Or the rules of signs for the tangent, secant, and cosecant, may be determined from those for the sine and cosine; as appears from the values given in Art. 8.

Some authors determine the diversity of sign for lines of the same denomination by their diametrically opposite position; but neither this proposition nor its converse is universally true.—(D'Alembert Opuscules, Tom. VIII. p. 271.)

Cagnoli adopts as a criterion of sign the established convention of analytic Geometry.

With respect to the criterion here adopted, Cagnoli remarks from Euler, that quantities do not change their signs unless in passing through infinity or cypher, but that the sign does not necessarily change in this passage.

11. Let us adopt a circle with radius unity (which we shall continue to do unless otherwise specified); and denote the circumference to this radius by the character 2π.

It is evident by taking $Ds = Am$, that As and Am have the same sine, viz. $\sin A = \sin(\pi - A)$

We have also $\sin(\pi + A) = \sin(2\pi - A.)$

All the arcs A, $2\pi + A$, $4\pi + A$, ... $2n\pi + A$ have the same sine, since they all commence at A and terminate at m. So also have the arcs $\pi - A$, $(2\pi + \pi) - A$, ... $(2n + 1)\pi - A$, $\therefore$ all arcs have precisely the same sine that are expressed by the two formulæ $2n\pi + A$, $(2n + 1)\pi - A$.

All arcs have the same *negative* sine that are included under the formulæ $(2n + 1)\pi + A$, $(2n + 2)\pi - A$.

All arcs with the same *positive* cosine are included in the formulæ $(2n\pi + A)$, $(2n+2)\pi - A$.

With the same *negative* cosine in the formulæ $(2n+1)\pi - A$, $(2n+1)\pi + A$.

All arcs with the same *positive* tangent in the formulæ $2n\pi + A$, $(2n+1)\pi + A$.

With the same *negative* tangent in the formulæ $(2n+1)\pi - A$, $(2n+2)\pi - A$.

All arcs with the same *positive* secant are included in the formulæ $(2n\pi + A)$, $(2n+2)\pi - A$.

With the same *negative* secant in the formulæ $(2n+1)\pi - A$, $(2n+2)\pi + A)$.

All arcs with the same *positive* cosecant are included in the formulæ $(2n\pi + A)$, $(2n+1)\pi - A$.

With the same *negative* cosecant in the formulæ $(2n+1)\pi + A$, $(2n+2)\pi - A$.

The versed sine is always positive, since the radius can never be less than the cosine.

Although the several arcs comprised in the formulæ $2n\pi + A$, $(2n+1)\pi - A$, have the same sine, still amongst the halves of these arcs there are two varieties of sine in magnitude and two in sign, ∴ the equation that expresses the sine or cosine of an arc in terms of the *sine* of its double must be a biquadratic soluble by the rules of quadratics.

Amongst the third parts of these arcs there are three varieties, which show that if we attempt the trisection of an angle by the sine we must fall on a cubic equation.

This is not the place, however, for such remarks, and we shall close them with proposing a problem of some interest.

"Given the sine of an arc, to find how many values the sine or cosine of its n^{th} part admits."

12. It remains to say something concerning the divisions of the circle that have been adopted.

The most usual and common division is into 360 parts, called degrees, each degree being subdivided into 90 parts, called minutes, and each minute into 90, called seconds, these again into thirds, &c. The French mathematicians, who have introduced a new system of weights and measures, have also introduced a new division of the circle. They have divided each quadrant into 100°, each degree into 100′, and each minute into 100″, &c. The chief advantage of this method is, that the minutes and seconds, &c. can be attached as decimals to the degrees, being 100th, 1000th, &c. parts thereof, *e. g.* $25° \, 15' \, 32''$ of French degrees, may be written $25°.1532$; $71° \, 1' \, 7''$, may be written $71°.0107$. This undoubtedly is a very considerable advantage in making computations of a complicated nature, and one should think ought to be a sufficient reason for its universal adoption. In order to be able to find the number of degrees in the English scale from those in the French, we have, if n be the number of French, $\frac{n.\,90}{100}$ for the number of English, or $n - \frac{n}{10}$. So that we need but, in the number of French, move the decimal point one place to the right and subtract the number so found from n. The decimals, if any in the result, are 10ths, 100ths, 1000ths, &c. of English degrees, and ∴ if multiplied by 6, 6×6, $6 \times 6 \times 6$, &c. becomes English minutes, seconds, thirds, &c. *E. g.* $26°.0735$ French give $23° \, 27' \, 58''.140$ English.

It is far simpler, however, to find either from the other, by knowing the value of a French degree, minute, or second in English measure; and of an English degree, minute, or second in French; as then these values can be repeated *ad libitum.*

French.	English.	French.	English.
$1° =$	$0° \, 54'$	$10° =$	$9°$
$1' =$	$0' \, 32''.4$	$10' =$	$5' \, 24''$
$1'' =$	0.324	$10'' =$	$3''.24$

Returning to our subject, it is now time to proceed to the formulæ for the values of Trigonometrical lines relative to the sum and difference of two arcs. Such formulæ are of the utmost importance, as they constitute the basis of the whole science of Trigonometry; and as they involve the principles already laid down, the reader, before proceeding farther, ought to take care that he become perfectly acquainted with these latter.

CHAP. II.

ON THE RELATIVE VALUES OF TRIGONOMETRICAL LINES BELONGING TO THE SUM OR DIFFERENCE OF TWO ARCS.

Let ACD (Fig. 2.) be a quadrant, from the centre C draw any two right lines Cm, Cn; from m let fall the perpendiculars mo, mr, ms; join sr, so, ro; it is obvious that the circle on Cm, as diameter, passes through the other three points s, r, o; and since the angle rCo is a rt. $\angle$ so must rso, and $\therefore ro = Cm = Cn$; and $\therefore$ the $\triangle\, ros$, which is obviously similar, is also equal to the $\triangle\, Cnp$; $\therefore$ $rs = Cp$, $so = np$.

Having thus prepared our construction, we have (Cor. 4. 16. 6. Elr.) the following Theorems:

$$Cm \times so = R \,.\, np = Cs \,.\, mo + Co \,.\, ms \qquad (1)$$
$$Cm \times rs = R \,.\, Cp = Cs \,.\, rm - ms \,.\, Cr \qquad (2)$$
$$sm \times ro = R \,.\, sm = so \,.\, rm - rs \,.\, mo \qquad (3)$$
$$Cs \times ro = R \,.\, Cs = Co \,.\, rs + Cr \,.\, mo \qquad (4)$$

If the arc Am be denoted by the character A, and mn by B; we have, using Trigonometrical language, Theorems (1) and (2) expressed as follows:

$$R \,.\, \sin\,(A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B \qquad (1)$$
$$R \,.\, \cos\,(A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B \qquad (2)$$

Denoting An by A, and Am by B; we have Theorems (3) and (4)

$$R \,.\, \sin\,(A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B \qquad (3)$$
$$R \,.\, \cos\,(A - B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B \qquad (4)$$

Thus we have the four fundamental formulæ of Trigonometry deduced from one Geometrical Theorem. In strictness, however, it is not necessary to deduce from the Theorem more than one of them, for instance the first, as the others can be easily formed from the first by analytical reasoning, as follows:

$\cos(A+B) = \sin\left(\frac{\pi}{2} - (A+B)\right) = \sin\left(\frac{\pi}{2} + A + B\right) =$ $\sin\left(\frac{\pi}{2} + A\right). \cos B + \cos\left(\frac{\pi}{2} + A\right). \sin B = \cos A. \cos B -$ $\sin A. \sin B.$

$\sin(A-B) = \cos\left(\frac{\pi}{2} - A + B\right) = \cos\left(\frac{\pi}{2} - A\right). \cos B -$ $\sin\left(\frac{\pi}{2} - A\right). \sin B = \sin A. \cos B - \cos A. \sin B.$

$\cos(A-B) = \sin\left(\frac{\pi}{2} - A + B\right) = \sin\left(\frac{\pi}{2} - A\right). \cos B +$ $\cos\left(\frac{\pi}{2} - A\right). \sin B = \cos A. \cos B + \sin A. \sin B.$

The reader will see, in an after part of this work, how all these formulæ might have been arrived at by analytical artifice, without any aid whatsoever from the elements of Geometry. The adoption of such a basis in an elementary work would be a most preposterous inversion of the natural order of instruction.

To find an expression for the tangent of the sum or difference of two arcs, we have $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$, (by formulæ (1) and (2),) $= \frac{\sin A. \cos B + \cos A. \sin B}{\cos A. \cos B - \sin A. \sin B}$, dividing each term of the numerator and denominator by $\cos A. \cos B$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A. \tan B} \qquad (5)$$

Similarly, $$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A. \tan B} \qquad (6)$$

Analogous formulæ arise from dividing (1) by (3) and (2) by (4), viz.

$$\frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\cot B + \cot A}{\cot B - \cot A} \qquad (7)$$

$$\frac{\cos(A+B)}{\cos(A-B)}=\frac{\cot B-\tan A}{\cot B+\tan A}=\frac{\cot A-\tan B}{\cot A+\tan B} \quad (8)$$

There are several formulæ that express values for the sine, cosine, or tangent of a single arc, A for instance. They can be had from the preceding formulæ, by using $\frac{1}{2}A$ for A and B.

From formula (1)...... $\sin A = 2.\sin\frac{A}{2}.\cos.\frac{A}{2}$ (*a*)

From .... (2) $\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2.\sin^2\frac{A}{2}$ (*b*)

From .... (5) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \dfrac{2\cot\frac{A}{2}}{\cot^2\frac{A}{2}-1}$ (*c*)

Whence by division $\tan A = \dfrac{2}{\cot\frac{A}{2}-\tan\frac{A}{2}}$ (*d*)

And hence again, $\cot\frac{1}{2}A - \tan\frac{1}{2}A = 2\cot A$ (*e*)

From formulæ (*a*) and (*b*) $\tan\frac{1}{2}A = \dfrac{1-\cos A}{\sin A}$ (*f*)

From the same,...... $\tan\frac{1}{2}A = \dfrac{\sin A}{1+\cos A}$ (*g*)

From (*f*) and (*g*) $\tan^2\frac{1}{2}A = \dfrac{1-\cos A}{1+\cos A}$ (*h*)

From (*h*) *comp.* and *div.* $\cos A = \dfrac{1-\tan^2\frac{1}{2}A}{1+\tan^2\frac{1}{2}A}$ (*i*)

From (*a*)......... $\sin A = \dfrac{2\tan\frac{1}{2}A}{1+\tan^2\frac{1}{2}A}$ (*k*)

From (*i*)......... $\cos A = \dfrac{\cot\frac{1}{2}A - \tan\frac{1}{2}A}{\cot\frac{1}{2}A + \tan\frac{1}{2}A}$ (*l*)

Multiply (*d*) by (*l*) $$\sin A = \frac{2}{\cot \frac{1}{2}A + \tan \frac{1}{2}A} \qquad (m)$$

From (*m*) and (*e*) $$\sin A = \frac{1}{\cot \frac{1}{2}A - \cot A} \qquad (n)$$

From the same... $$\sin A = \frac{1}{\cot A + \tan \frac{1}{2}A} \qquad (o)$$

From (*f*)......... $$\cos A = \frac{1}{1 + \tan \frac{1}{2}A . \tan A} \qquad (p)$$

Reverting to the formulæ for the sum and difference of two arcs, we find from (1) and (3) by addition and subtraction, and from (2) and (4), by the same process, four formulæ.

$$\sin (A + B) + \sin (A - B) = 2 \sin A . \cos B \qquad (9)$$

$$\sin (A + B) - \sin (A - B) = 2 \cos A . \sin B \qquad (10)$$

$$\cos (A + B) + \cos (A - B) = 2 \cos A . \cos B \qquad (11)$$

$$\cos (A - B) - \cos (A + B) = 2 \sin A . \sin B \qquad (12)$$

These formulæ may be exhibited in the following form, using $(n + 1) B$ for A.

$$\sin (n + 2) B + \sin n B = 2 \sin (n + 1) B . \cos B \qquad (13)$$

$$\sin (n + 2) B - \sin n B = 2 \cos (n + 1) B . \sin B \qquad (14)$$

$$\cos (n + 2) B + \cos n B = 2 \cos (n + 1) B . \cos B \qquad (15)$$

$$-\cos (n + 2) B + \cos n B = 2 \sin (n + 1) B . \sin B \qquad (16)$$

Writing A' and B' for $(A + B)$ and $(A - B)$ respectively, we have from formula (9), $\sin A' + \sin B' = 2 . \sin \frac{A' + B'}{2} . \cos \frac{A' - B'}{2}$, or restoring the former notation.

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) . \cos \tfrac{1}{2}(A - B \qquad (17)$$

Treating formulæ (10), (11), (12) after the same manner, we obtain

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) . \sin \tfrac{1}{2}(A - B) \qquad (18)$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) . \cos \tfrac{1}{2}(A - B) \qquad (19)$$

$$\cos B - \cos A = 2 . \sin \tfrac{1}{2}(A + B) . \sin \tfrac{1}{2}(A - B) \qquad (20)$$

Multiplying together (17) and (18), we have by formula (*a*)

$$\sin^2 A - \sin^2 B = \sin (A + B) . \sin (A - B) \qquad (21)$$

From (19) and (20),

$$\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos (A + B) \cos (A - B) \qquad (22)$$

From (17) and 18), $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} \qquad (23)$$

From (19) and (20), $$\frac{\cos A + \cos B}{\cos B - \cos A} = \frac{\cot \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} \qquad (24)$$

From (17) and (19),

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A + B) = \frac{\cos B - \cos A}{\sin A - \sin B} \qquad (25)$$

From (17) and (20),

$$\frac{\sin A + \sin B}{\cos B - \cos A} = \cot \tfrac{1}{2}(A - B) = \frac{\cos A + \cos B}{\sin A - \sin B} \qquad (26)$$

$$\tan A \pm \tan B = \frac{\sin (A \pm B)}{\cos A . \cos B} \qquad (27)$$

$$\cot B \pm \cot A = \frac{\sin (A \pm B)}{\sin A . \sin B} \qquad (28)$$

From (27), $$\tan^2 A - \tan^2 B = \frac{\sin (A + B) . \sin (A - B)}{\cos^2 A . \cos^2 B} \qquad (29)$$

From (28), $\cot^2 B - \cot^2 A = \dfrac{\sin(A+B).\sin(A-B)}{\sin^2 A.\sin^2 B}$ (30)

From (27), $\tan 2A - \tan A = \dfrac{\sin A}{2\cos^3 A - \cos A}$ (31)

From (13) we may form a table for particular cases of sines of multiple arcs, which we shall afterwards find of considerable use.

$$\left.\begin{aligned} \sin 2A &= 2.\sin A.\cos A \\ \sin 3A &= 3.\sin A - 4.\sin^3 A \\ \sin 4A &= (4\sin A - 8.\sin^3 A)\cos A \\ \sin 5A &= 5\sin A. - 20\sin^3 A + 16\sin^5 A. \\ \&c. &= \&c. \end{aligned}\right\} \qquad (A)$$

A similar table might be formed from (15) for the cosines.

$$\left.\begin{aligned} \cos 2A &= 2.\cos^2 A - 1 \\ \cos 3A &= 4.\cos^3 A - 3\cos A \\ \cos 4A &= 8\cos^4 A - 8\cos^2 A + 1. \\ \&c. &= \&c. \end{aligned}\right\} \qquad (B)$$

Several general series for sines and cosines of multiple arcs shall be given in the sequel of this work.

By the second formula in table (*B*), formula (31) is changed into

$$\tan 2A - \tan A = \frac{2.\sin A}{\cos A + \cos 3A} \qquad (32)$$

There is a particular class of *formulæ* useful as affording a means of passing from a Trigonometrical line computed for one arc to that for another, and useful also as affording means of verifying such results numerically computed.—Such are the following:

From (f) and (g) we have

$$\tan(45^\circ + \tfrac{1}{2}A) = \frac{1+\sin A}{\cos A} = \frac{\cos A}{1-\sin A} \qquad (f)$$

From the same, $\tan(45^\circ - \tfrac{1}{2}A) = \dfrac{1-\sin A}{\cos A} = \dfrac{\cos A}{1+\sin A}$ $\qquad (r)$

From (9),...... $\sin(60^\circ + A) - \sin(60^\circ - A) = \sin A$ $\qquad (s)$

From (11),... $\cos(60 + A) + \cos(60 - A) = \cos A$ $\qquad (t)$

From (1), (2), (5), and (6), using 45° for A we have four formulæ.

$$\cos(45^\circ - B) = \sin(45^\circ + B) = \frac{\cos B + \sin B}{\sqrt{2}} \qquad (u)$$

$$\cos(45^\circ + B) = \sin(45^\circ - B) = \frac{\cos B - \sin B}{\sqrt{2}} \qquad (v)$$

$$\tan(45^\circ + B = \frac{1+\tan B}{1-\tan B} \qquad (w)$$

$$\tan(45^\circ - B) = \frac{1-\tan B}{1+\tan B} \qquad (x)$$

From (w) and (x)

$$\tan(45^\circ + B) - \tan(45^\circ - B) = \frac{4.\tan B}{1-\tan^2 B} = 2\tan 2B \qquad (y)$$

From squaring (u),

$$2.\sin^2(45^\circ + B) = 2.\cos^2(45^\circ - B) = 1 + \sin 2B \qquad (z)$$

Hence making B negative,

$$2.\sin^2(45^\circ - B) = 2\cos^2(45 + B) = 1 - \sin 2B \qquad (\beta)$$

The line $B\,o$ in (Fig. 1.) being called the coversed-sine, we have here expressions for it and $o\,E$, the arc $A\,m$ being denoted by $2\,B$.

From (z) and (β) $\dfrac{1+\sin 2B}{1-\sin 2B} = \tan^2(45^\circ + B)$ $\qquad (\gamma)$

$$\frac{1 - \sin 2B}{1 + \sin 2B} = \tan^2(45^\circ - B) \qquad (\delta)$$

Hence $\sin 2B = \dfrac{\tan^2(45^\circ + B) - 1}{\tan^2(45 - B) + 1} = \dfrac{1 - \tan^2(45 - B)}{1 + \tan^2(45^\circ - B)}$ (ϵ)

or, $\sin 2B = \dfrac{\tan(45^\circ + \frac{1}{2}B) - \tan(45^\circ - \frac{1}{2}B)}{\tan(45^\circ + \frac{1}{2}B) + \tan(45^\circ - \frac{1}{2}B)}$ (η)

Dividing (η) by (y) $\cos 2B = \dfrac{2}{\tan(45^\circ + B) + \tan(45^\circ - B)}$ (θ)

From (u) and (v) $\cos 2B = 2.\cos(45^\circ + B).\cos(45^\circ - B)$ (λ)

If $A = 30^\circ$ we have from (9) $\sin(30^\circ + B) - \sin(30^\circ - B) = \sin B.\sqrt{3}$ (μ)

From (10)......... $\cos(30^\circ + B) + \cos(30^\circ - B) = \cos B\sqrt{3}$ (ν)

If $A + B + C = 180^\circ$ then $\tan C = -\tan(A+B) = \dfrac{\tan A + \tan B}{\tan A.\tan B - 1}$

whence we have in this case $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$.

This singular result shews us that the question, *to find three numbers whose sum is equal to their product*, has an infinite number of solutions.

Expressions for the tangents of the sums of arcs, may readily be formed from the value already given for $\tan(A + B)$, as follows:

$\tan(A + A' + A'') = \dfrac{\tan(A + A') + \tan A''}{1 - \tan(A + A')\tan A''}$, but denoting the successive tangents by $t\,t'\,t''$, &c. t^{n-1}, $\tan(A + A') = \dfrac{t + t'}{1 - tt'}$ $\therefore$ we have $\tan(A + A' + A'') = \dfrac{t + t' + t'' - tt't''}{1 - (tt' + tt'' + t't'')}$. By pursuing a similar track, $\tan(A + A' + A'' + A''') = \dfrac{\underset{4}{\Sigma}.t - \underset{4}{\Sigma}.tt't''}{1 - \underset{4}{\Sigma}.tt' + \underset{4}{\Sigma}.tt't''t'''}$

and in general, $\tan((n)\text{ arcs}) = \dfrac{\Sigma_n.\, t - \Sigma_n(tt't'') + \Sigma_n(tt't''t'''t^{\text{iv}})}{1 - \Sigma_n(tt') + \Sigma_n.(tt't''t''') - \&c. +}$. (π)*

Now if this be true for n arcs, we can prove it true for $(n+1)$ arcs, for $\tan(n+1)$ arcs $= \dfrac{\tan(n)\text{ arcs} + t_n}{1 - \tan(n)\text{ arcs}.\, t_n}$, in which, substituting the above expression for tan (n) arcs, and multiplying above and below by the denominator in the expression for the tan (n) arcs, we obtain in the numerator of tan ($n+1$) arcs all the new odd combinations, and in the denominator all the new even ones that will suffice to represent the expression for tan ($n+1$) arcs under a form precisely similar to that for tan (n) arcs.

Thus we have shewn this series to be universally true by an argument such as the following; if true for (n) arcs it is true for ($n+1$) arcs, and by the same reasoning, if true for ($n+1$) true for ($n+2$), &c.; but it is true if $n=2$, $\therefore$ if $n=3$, $n=4$, &c. This mode of arguing, though ultimately founded on induction, ceases to be induction when verified as above, and, in fact, ranks with the highest species of mathematical evidence. Singularly enough this mode of reasoning has been passed over altogether by those metaphysical writers who have treated on induction.

If $A = A' = A'' =$, &c. in the above series, we obtain an expression for $\tan nA$. Recollecting that the combinations of n things in *ones* $= n$; in *threes* $= n.\dfrac{n-1}{2}.\dfrac{n-2}{3}$; in *fives* $= n.\dfrac{n-1}{2}.\dfrac{n-2}{3}.\dfrac{n-3}{4}.\dfrac{n-4}{5}$, &c.; in *pairs* $= n.\dfrac{n-1}{2}$; in *fours* $= n.\dfrac{n-1}{2}.\dfrac{n-2}{3}.\dfrac{n-3}{4}$, &c. we have

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* The character Σ is not a factor; it merely denotes an assembling or grouping together of like quantities: for instance, $\Sigma_5(tt't'')$ denotes the sum of the several combinations in *threes* of 5 tangents.

$$\tan n A = \frac{n . \tan A - n . \frac{n-1}{2} . \frac{n-2}{3} . \tan^3 A + \&c.}{1 - n . \frac{n-1}{2} . \tan^2 A + n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . \tan^4 A - \&c.} \quad (\varrho)$$

All these expressions for tangents of multiple arcs were first arrived at by De Lagny, in the Memoirs of the Academy of Sciences for 1705. He obtained them by induction, but did not verify it as above.

Multiplying series (ϱ), above and below, by $\cos^n A$, we obtain

$$\frac{\sin n A}{\cos n A} = \frac{n . \cos^{n-1} A . \sin A - n . \frac{n-1}{2} . \frac{n-2}{3} . \cos^{n-3} A . \sin^3 A + \&c.}{\cos^n A - n . \frac{n-1}{2} . \cos^{n-2} A . \sin^2 A + \&c.} \quad (\sigma)$$

The numerator and denominator of this series might be shewn separately equal to $\sin n A$ and $\cos n A$, but as we shall obtain all these series afterwards in a more satisfactory manner, it does not seem necessary at present to enter upon this subject.

The series for the sines and cosines, or rather the chords, of multiple arcs, were first arrived at by Vieta. The same subject was afterwards treated of by Oughtred and Wallis, and subsequently by Bernouilli and Herman. De Lagny was the first who arrived at expressions for secants and tangents of multiple arcs.

Having thus gone through the fundamental formulæ of Trigonometry, we shall next proceed to their most important application, viz. the calculation of triangles, both plane and spherical. The student is not to consider that the formulæ obtained are merely an exercise in elementary analytical reasoning; as such certainly they are of considerable use, but of infinitely greater as putting the reader in possession of instruments of calculation, without which his progress would be necessarily limited and obscure. The justice of this remark will probably be more readily acknowledged after proceeding farther in this work.

CHAP. III.

ON THE RESOLUTION OF PLANE TRIANGLES.

1. HAVING now to pass to one of the applications of the preceding Theory, which is of primary importance, viz. the resolution of plane triangles, we shall introduce a second Geometrical Theorem, to serve as a connecting link between the Theory and this application of it. There are several Geometrical Theorems, any one of which might be used on this occasion, but for a particular reason we shall select that which gives the cosine of an angle of a triangle in terms of the sides.

2. In figure (3), let the sides of the $\triangle$ be denoted by the characters a, b, c, and the angles respectively opposite by A, B, C, letting fall a perpendicular from C on c, and denoting the distance between the angle A and the foot of the perpendicular by p, we have (13. 2. Elr.) $a^2 = b^2 + c^2 - 2cp$, $\therefore p = \frac{b^2 + c^2 - a^2}{2c}$, but p is the cosine of A if b be the radius, and $\therefore$ to radius unity $p = b \cos A$, $\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Similarly $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$; and $\cos C = \frac{b^2 + a^2 - c^2}{2ab}$.

If A be obtuse the formula does not change, for then the signs both of p and $\cos A$ change, and $\therefore$ the sign of the result is not changed.

3. We have $1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a + c - b)}{2bc}$, or denoting the semi-sum of the three sides by s; $1 - \cos A = \frac{2(s - c)(s - b)}{bc}$.

4. We have also $(1+\cos A)=\frac{(b+c)^2-a^2}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc}=\frac{2.s.(s-a)}{bc}$.

5. Multiplying the values for $(1-\cos A)$ and $(1+\cos A)$, we have $1-\cos^2 A=\sin^2 A=\frac{4}{b^2 c^2}\cdot\left(s.(s-a)\ (s-b)\ (s-c)\right)$,

$\therefore\ \sin A=\frac{2}{bc}\cdot\ \sqrt{s.(s-a)(s-b)(s-c)}$.

6. The area of the $\triangle=\frac{cp}{2}=\frac{c.b.\sin A}{2}=\sqrt{s.(s-a)(s-b)(s-c)}$; which is an expression for the area that might easily be arrived at by plane Geometry.

Denoting the double radical by the character N, we have $\sin A=\frac{N}{bc}$, and $\sin B=\frac{N}{ac}$, $\therefore\ \frac{\sin A}{\sin B}=\frac{a}{b}$, or the sines of the angles of a $\triangle$ are as the sides respectively opposite.

Let us proceed to the solution of triangles, enumerating the various cases. In a $\triangle$ there are six quantities, any three of which being given, the remaining quantities may be found, unless the *data* be the three angles.

Cases of right angled triangles, C *being the right angle.*

First Case.—Given a, b, to find c and A, B.

$\frac{a}{b}=\frac{\sin A}{\sin B}=\frac{\sin A}{\sin\left(\frac{\pi}{2}-A\right)}=\tan A$; and thus having A, we have $B=90°-A$; $c=b.\sec.A$, or $c=\sqrt{a^2+b^2}$.

These formulæ in logarithms, give log. tan $A=$ log. $r+$log. $a-$ log. b; where r is the radius of the tables; log. $c=$log. $r+$ log. $b-$ log. cos A.

Second Case.—Given c, B, to find a, b, A.

$b=c.\sin B$; $A=90-B$; $a=c.\sin A$ or $a=\sqrt{c^2-b^2}$.

In logarithms, log. b = log. c + log. sin B — log. r; log. a = $\frac{1}{2}\left\{\text{log.}(c+b)+\text{log.}(c-b)\right\}$

Third Case.—Given b, A, to find a, c, B.

$$a = b.\ \tan A;\ c = \frac{b}{\cos A};\ B = 90 - A.$$

In logs. log. a = log. b + log. tan A—log. r; log. c = log. b — log. cos A + log. r.

N. B.—The tabular radius is taken 10^{10} $\therefore$ log. $r = 10$.

Cases of oblique-angled triangles.

First Case.—Given A, B, a, to find b, c, C.

$$C = 180^\circ - (A + B);\ b = a.\ \frac{\sin B}{\sin A};\ c = a.\ \frac{\sin C}{\sin A}.$$

In logs. log. b = log. a + log. sin B — log. sin A. Similarly for c.

Second Case.—Given a, b, B; to find A, C, c.

$\sin A = \frac{a}{b}.\ \sin B;\ C = 180^\circ - (A + B);\ c = b.\ \frac{\sin C}{\sin B}$ or $c = a.\ \cos B \pm \sqrt{b^2 - a^2.\ \sin^2 B}$.

This formula for c, however, is not adapted to logarithmic computation. It is manifest from the figure in this case, that if $b > a$, from knowing a, b, and B, we cannot know whether we have found A or $180^\circ - A$, but if $a < b$, the angle A must be acute.—*Cagnoli*, p. 132.

In logs. log. sin A = log. a + log. sin B — log. b
log. c = log. b + log. sin C — log. sin B.

The negative radical in the second value of c answers to the case when A is obtuse.

Third Case.—Given a, b, C, required A, B, c.

1° We have $\frac{a}{b}=\frac{\sin A}{\sin B}$, $\therefore$ *comp.* and *div.* $\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}$

or by formula (23) chap 2, $=\dfrac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$, or *the sum of the two sides : their difference :: tan. ½ sum of base angles : tan ½ diff.*

We have also the two following formulæ, to which we shall meet analogous ones in spherical Trigonometry.

$$\tan\tfrac{1}{2}(A-B)=\cot\tfrac{1}{2}C.\frac{a-b}{a+b},\ \cot\tfrac{1}{2}(A-B)=\cot.\tfrac{1}{2}C.\frac{a+b}{a-b}$$

2° To find the third side, c, we have $c^2=(a^2+b^2-2ab\cos.C)$: but this requires to be adapted to logarithmic computation. In order to this we may write c^2 in several forms, viz.

$$c^2=\left\{(a+b)^2-2ab(1+\cos C)\right\}=\left\{(a-b)^2-2ab\times(\text{versed}\sin C)\right\}$$

$$\text{or } c=(a+b)\sqrt{1-\frac{4ab}{(a+b)^2}.\cos^2\frac{C}{2}}=(a-b)\sqrt{1+\frac{2ab}{(a-b)^2}.\text{versed}\sin C}$$

$=(a-b)\sqrt{1+\frac{4\,ab}{(a-b)^2}.\sin^2\frac{C}{2}}$. In the first value, the quantity $\frac{4\,ab}{(a+b)^2}.\cos^2\frac{C}{2}$ may be computed logarithmically; and, since it is a proper fraction, it may be taken as the sine squared of an auxiliary arc θ, and using then this form, we have $c=(a+b)\cos\theta$. Denoting the similar quantity in the second value of c by $\tan^2\theta$ we have $c=(a-b)\sec.\theta$.

We shall afterwards give two very elegant series, one for discovering a base angle, and the other the third side in this case, taken from the Trigonometry of Legendre.

Fourth Case.—Given a, b, c, to find A, B, C.

We have already found $\sin A=\frac{2}{bc}\sqrt{s.s-a.s-b.s-c}$ (1)

$$\sqrt{\tfrac{1}{2}(1+\cos A)}=\cos\frac{A}{2}=\frac{\sqrt{s.\,s-a}}{\sqrt{bc}} \qquad (2)$$

$$\sqrt{\tfrac{1}{2}(1-\cos A)}=\sin\frac{A}{2}=\frac{\sqrt{s-b.\,s-c}}{\sqrt{bc}} \qquad (3)$$

And thus we have from (2) and (3), $\tan\frac{A}{2}=\sqrt{\frac{s-b.\,s-c}{s.\,s-a}}$ (4)

Similar formulæ may of course be used for finding B and C.

Recollecting in all these cases that the angles are the objects of investigation, and that they are to be had by their sines, cosines, tangents, &c.; the utility of several formulæ for the same case is instantly obvious. In such cases approximate values are only to be aimed at, and the more rapid the approximation, of course so much the more convenient; and hence it is desirable that a small error on the sine, cosine, or tangent, should entail the smallest possible error on the angle. By the principles of the differential calculus we have $dA : d.\sin A :: 1 : \cos A$; hence formula (1) in the fourth case ought not to be used for arcs nearly 90°, but is conveniently applicable to arcs under 45°. By the same calculus, $dA : d.\cos A :: 1 : \sin A$, hence formula (2) ought to be used for arcs nearly (90°), but < 90 and formula (3) for arcs nearly (90°), but $> 90°$. Again, $dA : d\tan A :: \sec^2 A : 1$, $\therefore$ formula (4) should not be used for arcs nearly 180°, but formulæ (1) and (2) are then convenient.

Hence different formulæ for the same purpose, though certainly useless in a Theoretic point of view, add very much to the convenience and expedition of practice. The reader cannot mistake this subject, if he take the pains to read all Woodhouse has said on it from page 91 to 97 of his Trigonometry. It may be worthy of remark, that the formulæ of right angled triangles might have been deduced from these found for oblique, by supposing in the latter $C=90°$.

The principles laid down in this and the preceding chapter might be successfully applied to the solution of Geometrical problems. Previously to giving numerical examples, it would be advisable to shew some instances of this application. This we shall accordingly proceed to do in some of the more difficult cases of this species of questions.

1. *To draw a line through a point within the legs of a right angle, such that its length be a minimum.*

Let p, p', be the perp. from the point on the legs, and θ the angle the required line makes with p, and therefore $90° - \theta$ the angle it makes with p', then $\frac{p}{\cos\theta} + \frac{p'}{\sin\theta}$ is the length of the line. Differentiating and equating the differential to cypher, we have $\tan^3\theta = \frac{p'}{p}$.

If r, r', be the portions on the legs between the feet of the perpendiculars p, p', and the ends of the line; it is obvious that $\frac{r}{p} = \tan\theta = \sqrt[3]{\frac{p'}{p}}$, $\therefore r^3 = p^2 . p'$ whence r is the first of two means between p and p', and $\therefore$ r and r' the two means.

Hence the line drawn according to Philo's method of finding two means is the *minimum* line.

2. *Given the base difference of base angles and rectangle of sides, to find the triangle.*

Let a be the base, then one side $= a . \frac{\sin C}{\sin A}$, and the other $= a . \frac{\sin B}{\sin A}$, and by the question $\frac{a^2 . \sin B . \sin C}{\sin^2 A} = m^2$ $\therefore$ $\cos(B - C) - \cos(B + C) = \frac{2m^2}{a^2} . \sin^2 A$, but $\cos(B + C) = -\cos A$, and let $\cos(B - C) = t$, then $\cos A = \frac{2m^2}{a^2} . \sin^2 A - t$, $\therefore$ $\cos^2 A + \frac{a^2}{2m^2} . \cos A = 1 - \frac{t a^2}{2 m^2}$; whence we have $\cos A$.

In Fig. (4), let one of the triangles given by this solution be ABC, draw Bm to bisect the vertical angle, draw mm' the diameter of the circumscribing circle, draw $m'B'$ parallel to mB; then the second triangle required is $AB'C$.

3. *To inscribe in a circle a triangle, such that its sides shall have a given ratio, and the rectangle of the segments of the base by the perpendicular shall be a maximum.*

Let A, B, be the angles opposite the sides whose ratio is given, and d the diameter of the circle. Then one segment of the base $= d.\sin A.\cos B$, and the other $= d.\cos A.\sin B$, $\therefore$ $\sin A.\cos A.\sin B.\cos B$, must be a *maximum*; hence we have $\frac{dA}{dB} = -\frac{\sin 2A.\cos 2B}{\sin 2B.\cos 2A}$. Let m be the exponent of the given ratio, then $\sin A = m.\sin B$, $\therefore \frac{dA}{dB} = m.\frac{\cos B}{\cos A} = \frac{\sin A.\cos B}{\sin B.\cos A}$, $\therefore$ we have, $-\frac{\tan 2A}{\tan 2B} = \frac{\tan A}{\tan B}$. In this expression, substituting for $\tan 2A$ and $\tan 2B$ their values in terms of $\tan A$, $\tan B$, we find $\tan^2 A + \tan^2 B = 2$.

This last result shews a beautiful property of the required triangle. Combining this with the equation $\sin A = m.\sin B$, we can find $\sin A$ and $\sin B$.

This solution, which is due to Mr. Lowry, the reader will find in a late number of Leybourne's Repository.

4. *Given the external and internal bisectors of the vertical angle of a triangle, and the difference of the sides, to find the triangle.*

Let b be the internal bisector, and b' the external; let A denote the angle contained by b, and the hypothenuse of the right angled triangle, that has b and b' for its sides, then A is given, for $\sin A = \frac{b}{\sqrt{b^2 + b'^2}}$. Let B denote half the vertical angle, and one side $= \frac{b.\sin A}{\sin(A-B)}$, and the other $= \frac{b.\sin A}{\sin(A+B)}$, let then $\frac{b.\sin A}{\sin(A-B)} - \frac{b.\sin A}{\sin(A+B)} = m$. Arranging the terms we have from this $b.\sin A.\left\{\frac{\sin(A+B) - \sin(A-B)}{\sin(A+B).\sin(A-B)}\right\} = m$, and substituting in the numerator and denominator by formulæ (9) and (21) Chap. 2. we obtain $\frac{b.\sin 2A.\sin B}{\sin^2 A - \sin^2 B} = m$, whence by solution we find $\sin B = \frac{\sin A}{m}\left\{-b.\cos A \pm \sqrt{b^2.\cos^2 A + m^2}\right\}$.

5. *Given the three sides of a triangle, to find its surface, the radius of the inscribed circle, and that of the circumscribed circle:*

An expression for the area has been already arrived at in the commencement of this chapter, viz. $A' = \sqrt{s.\overline{s-a}.\overline{s-b}.\overline{s-c}}$.

By cor. 2. 16. 6, Elr. Euclid, the radius of the circumscribed circle, which we shall call R, $= \frac{ac}{2p}$, p being the perp. from B on the side b, $\therefore R = \frac{abc}{2pb}$, but $pb = 2A'$, $\therefore R = \frac{abc}{4.\sqrt{s.\overline{s-a}.\overline{s-b}.\overline{s-c}}}$.

Let r be the radius of the inscribed circle, then $sr = A'$, $\therefore$ we have $r = \sqrt{\frac{\overline{s-a}.\overline{s-b}.\overline{s-c}}{s}}$.

6. *Being given the four sides of an inscribed quadrilateral, to find the radius of the circle, the surface of the quadrilateral, and the angles.*

Let a, c, and b, d, be sides respectively opposite, and x, y, the two diagonals, then $xy = ac + bd$ (cor. 4. 16. 6. Elr.). Since the opposite angles of the quadrilateral are supplemental, denoting the angle between a and d by θ, we have $(ad+bc). \sin\theta = 2A'$, and the angle between a, b, by θ', we have $(ab+cd). \sin\theta' = 2A'$, but $\frac{x}{y} = \frac{\sin\theta'}{\sin\theta} = \frac{ad+bc}{ab+cd} \therefore$ we have $x = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$, and $y = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$. By the preceding problem the radius of the circle circumscribed to the triangle that has a, b, x, for its sides, may be expressed by the formula, $R = \frac{abx}{4.\sqrt{s.(s-a)(s-b)(s-x)}}$ substituting in this the value previously found for x, and decomposing into factors

$$R = \sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)}}$$

By the preceding the area of the triangle that has a, b, x, for its sides $= \frac{a.b.x.}{4R}$. In like manner the area of the triangle that has

c, d, x, for its sides $=\frac{cdx}{4R}$, $\therefore$ the area of the quadrilateral $=\frac{(ab+cd)x}{4R}=\frac{1}{4}\cdot\sqrt{(a+b+c-d).(a+b+d-c).(a+c+d-b).(b+c+d-a)}$ or if $p=\frac{a+b+c+d}{2}=\sqrt{p-a.p-b.p-c.p-d}$.

To find an angle, for instance, the angle between a and b. Denote it by B; then $\cos B=\frac{a^2+b^2-x^2}{2ab}=\frac{a^2+b^2-c^2-d^2}{2ab+2cd}$ by introducing its value for x^2. Hence we have $\frac{1-\cos B}{1+\cos B}=\frac{(c-a)^2-(a-b^2)}{(a+b)^2-(c-d)^2}$, or since $\frac{1-\cos B}{1+\cos B}=\tan^2\frac{1}{2}B$; we have, decomposing the preceding result into factors, $\tan^2\frac{1}{2}B=\frac{(a+c+d-b)(b+c+d-a)}{(a+b+c-d)(a+b+d-c)}$, or $\tan\frac{1}{2}B=\sqrt{\frac{(p-a).(p-b)}{(p-c).(p-d)}}$. The opposite angle B' being supplemental to B, we have $\tan\frac{1}{2}B'=\cot\frac{1}{2}B=\sqrt{\frac{(p-c).(p-d)}{(p-a).(p-b)}}$. The adjoining angle between a and d being called C, we have similarly $\tan\frac{1}{2}C=\sqrt{\frac{(p-a).(p-d)}{(p-b).(p-c)}}$ and $\tan\frac{1}{2}C'=\sqrt{\frac{(p-b).(p-c)}{(p-a).(p-d)}}$.

This exercise of Trigonometrical artifice might be continued to a far greater extent. As such a proceeding, however, would not be very consistent with the plan of an Elementary Treatise on Trigonometry, the further prosecution of this subject is left to the reader's sagacity.

We shall now proceed to give some numerical illustration of the solution of cases of plane triangles. In this we shall of course make use of the tables that have been computed for logarithmic calculation, and by doing so we shall be obliged to use a part of our Theory, the discussion of which cannot be given until the second part of this treatise. This anticipation is unavoidable when practice is introduced, as in it we must have recourse to all the aid that Theory can afford.

Examples of the resolution of right-lined triangles.

Given in a rt. $\angle d$ plane triangle $a = 197.3$, $B = 51°56'$ and $\therefore A = 38° 4'$; we have $b = a. \frac{\sin B}{\sin A}$, $\therefore$ log. $b =$ log. $a +$ log. sin B — log. sin A, or taking the values from the tables,

$$\begin{array}{lll} \text{log. } b & = & 2.2951271 \\ \text{log. sin } B & = & 9.8961369 \\ \hline \text{Sum} & = & 12.1912640 \\ \text{log. sin } A & = & 9.7899880 \\ \hline \text{diff.} & = & 2.4012760 = \text{log. } b, \therefore b = 251.9278. \end{array}$$

This example has been solved from the tables of Hutton, *fifth edition.* These tables, as well as those of Sherwin and Taylor, are computed but to seven places of figures, so that should there be two angles, the logs. of whose sines approached so near equality as not to differ till after the seventh figure, it would be impossible, knowing the sine, to know which angle ought to be adopted as the true value. *E. g.* In the tables of Hutton, the log. of the sine of 89° 57′ is 9,9999998, and the same, as appears by adding the *proportional part* in the column of differences, is the log. of the sines of all arcs up to 89° 57′ 50″, so that we cannot, with such tables, if we use the value of the *sine* in computing such large arcs, pronounce upon being accurate to nearer than 1′.

In Vlacq's tables, published in 1633, we have

Arcs.	Logs. sines.
89° 57′	9.9999998346
89° 57′ 10″	9.9999998525
89° 57′ 20″	9.9999998693
89° 57′ 30″	9.9999998851
89° 57′ 40″	9.9999998999
89° 57′ 50″	9.9999999137

This comparative imperfection of the modern tables is perhaps more apparent than real, for it is never necessary to find such large

arcs from their sines, as Trigonometry is able to furnish not merely adequate solutions but convenient and concise ones.

Example 2.—Given $b = 1123.7943$, $B = 61° 40' 12''$, $A = 28° 19' 48''$, to find the hypothenuse c.

$$C. \sin B = b, \therefore \log. c. = \log. b - \log. \sin B + 10.$$

$$\begin{array}{lcl} \log. b & = & 3.0506868249 \\ \log. r. & = & 10 \\ \hline \text{Sum} & = & 13.0506868249 \\ \log. \sin B & = & 9.9445956729 \\ \hline \log. c. = \text{diff.} & = & 3.1060911520 \end{array}$$

Hence we obtain $c = 1276.7067$ f. p.

This is an example computed by the immense and very accurate tables of Vlacq. Let us take another example for a different case from the same tables.

Example 3.—The most interesting and useful case of plane triangles is when we are given two sides and the included angle, to find the base angles. Taking the instance of this that Vlacq gives in the first book of his *Trigonometria Artificialis* prefixed to his tables, we have

$$\begin{aligned} a &= 1276.7067 \\ b &= 631.5525 \\ C &= 37° 26' 43'' \end{aligned}$$

$$\begin{aligned} \therefore a + b &= 1908.2592 \\ a - b &= 645.1542 \\ \frac{A+B}{2} &= 71° 16' 38'' \end{aligned}$$

But by the formula for this case, $\log. \tan \frac{A-B}{2} = \log. \tan \frac{A+B}{2} + \log. (a - b) - \log. (a + b)$

$$\begin{array}{lll} \text{and log. tan } 71^\circ\,16'\,38'' & = & 10.4698970632 \\ \text{log. } 645.1542 & = & 2.8096635289 \\ \hline \text{Sum,} & = & 13.2795605921 \\ \text{log. } 1908.2592 & = & 3.2806373647 \\ \hline \text{Diff.} & = & 9.9989232274 = \text{log. } \dfrac{A-B}{2} \end{array}$$

and the angle answering this logarithm is $44^\circ\,55'\,43''\,51'''$. Hence

$$A = 71^\circ\,16'\,38'' + 44^\circ\,55'\,43''\,51''' = 116^\circ\,12'\,21''\,51'''$$

$$B = 71^\circ\,16'\,38'' - 44^\circ\,55'\,43''\,51''' = 26^\circ\,20'\,54''\,9'''$$

Instead of going more diffusely into such calculations, it is perhaps better to offer to the reader several examples with their answers, leaving to himself the operations requisite for the purpose of arriving at the results, as indeed it is only by this mode of proceeding that he can become expert in the use of the tables.

1° To compute the formula $\tan \frac{1}{2}A = \sqrt{\dfrac{s-b.\,s-c}{s.\,s-a}}$ when $c = 1276.7067$; $a = 865.1765$; $b = 631.5525$,

$$R. \quad \tfrac{1}{2}A = 58^\circ\,6'\,11''\,30'''.$$

2° To compute the formula $a = b\,\dfrac{\sin A}{\sin B}$, when $b = 1276.7067$; $A = 37^\circ\,26'\,43''\,30'''$; $B = 63^\circ\,47'\,37''$,

$$R. \quad a = 865.1765.$$

3° To compute the formula $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \tan \dfrac{A+B}{2}$, when $a = 562$; $b = 320$; $C = 128^\circ\,4'$

$$R. \quad A = 33^\circ\,34'\,40''.\ B = 18^\circ\,21'\,20''.$$

4° To compute the formula $\sin A = \dfrac{2}{bc}\sqrt{s.(s-a)\,(s-b)\,(s-c)}$ when $a = 33$; $b = 42.6$; $c = 53.6$,

R. The radical, or area of the $\triangle = 702.868$,

and $A = 37° 59' 53''$. If B and C be similarly computed we find $B = 52° 37' 46'' 28'''$; $C = 89° 22' 20'' 32'''$.

N. B.—We may verify by adding the three angles.

The angle C in this case, if computed by the common tables, may be $89° 22' 21''$, $89° 22' 25''$, or any angle between. The result above given has been obtained from the tables of Vlacq.

If the angle C had been computed in this case by the formula $\tan \frac{C}{2} = \sqrt{\frac{s-a.\,s-b}{s.\,s-c}}$, the result obtained by the common tables would be $C = 89° 22' 20''.\frac{1}{4}\frac{1}{4}$. This result, obtained by a different formula more accurately, shews in the most striking light the advantage of several formulæ for the same case.

5° If one angle of a plane $\triangle$ be $139° 54'$, and the sides about it 5 and 9, required the other angles.

R. $26° 0' 10'' 36'''$ and $14° 5' 49'' 24'''$.

6° In a right angled triangle, given a side $a = 1123.7943$ and the angle opposite, viz. $A = 61° 40' 12''$, required the hypothenuse,

R. $c = 1276.7067$.

CHAP. IV.

ELEMENTARY THEORY OF PLANES AND SOLID ANGLES—THE SPHERE AND SPHERICAL TRIANGLES—DEDUCTION OF FUNDAMENTAL FORMULÆ PREPARATORY TO THE ANALYTIC THEORY OF SUCH TRIANGLES.

A right line is perpendicular to a plane when it is perpendicular to all the lines that meet it in that plane.

A right line is parallel to a plane, when they cannot meet, though indefinitely produced.

Two intersecting right lines *AB*, *AC*, determine a plane; for conceive any plane containing *AB*, to turn on *AB* until it passes through *C*, then it will contain the points *A* and *C*, and ∴ the line *AC*, and thus become determined.

Two parallel right lines determine a plane; for the plane of a common intersecting line and one of them also contains the other.

The intersection of two planes is necessarily a right line; for if any three points of it formed a triangle, each of the planes would become the plane of this triangle, and ∴ they would coincide.

If a right line be perpendicular to two right lines meeting it in a plane, at an angle $< 180°$, it is perpendicular to every line meeting it in the same plane. "Let *oh* (Fig. 5.) be perpendicular to two lines *om*, *on*, meeting it at *o*, let *os* be any other right line from the same point. Through any point of it, *s*, draw a right line, *nsm*, so that $ns=sm$ join *hm*, *hs*, *hn*, then $hm^2+hn^2=2.hs^2+2.sn^2$, but $hn^2=ho^2+on^2$ and $hm^2=ho^2+om^2$ (*by hypothesis*), ∴ $2.ho^2+om^2+on^2=2.hs^2+2.sn^2$, and again $om^2+on^2=2.os^2+2.sn^2$. Subtracting these two last equations, arranging and dividing by 2, we have $hs^2=ho^2+os^2$ ∴ the angle *hos* is a right angle."

This proof shews the legitimacy of the definition that has been given of a line perpendicular to a plane.

There is hence evidently but one line perpendicular to the same plane at the same point.

The perpendicular is the shortest line from a point on a plane. This is evident from what has been said.

The inclination of an oblique line to a plane is the complement of its inclination to the perpendicular from any point in it.

If two angles, not in the same plane, have their sides parallel and in the same direction, the angles will be equal. "This is evident by taking equal portions on the legs of the angles and joining them, the joining lines are equal and parallel, as being equal and parallel to the lines joining the vertices of the angles, ∴ the lines in the planes of the angles joining their extremities are

equal. We have then two triangles with the three sides respectively equal, $\therefore$ the vertical angles are equal."

The measure of the inclination of two planes is the angle contained between lines in those planes perpendicular to their common intersection. To prove the legitimacy of this measure it is necessary to prove,

1° That it is the same for all the points of the line of intersection. This is evident from the fact that each line moves parallel to itself.

2° That if the angle of the planes increase or diminish in a certain ratio, the angle of the perpendiculars will increase or diminish in the same ratio. "This is evident if we consider a circular arc connecting equidistant points on the perpendiculars."

It is the same with angles formed by two planes as with angles formed by two right lines; when the two planes cut, the angles vertically opposite are equal, and the adjacent angles together equal to two right angles.

If a line (l) be perpendicular to a plane (A), every plane (B) containing the line will be perpendicular to the same plane (A). "For in the plane (A) draw a line (l') perpendicular to the intersection of the planes (A) and (B), then the line (l) is perpendicular to (l'), and each of them to the intersection; since they are perpendicular to the intersection, they have the inclination of the planes in which they lie, and since they are mutually perpendicular the planes (A) and (B) are at right angles."

A solid angle is the angular space comprised between several planes that unite in the same point. For forming such there are required at least three planes.

If a solid angle be formed by three plane angles, the sum of any two of them is greater than the third. "Let S (Fig. 6.) be a solid angle contained by the planes ASB, BSC, CSA, and suppose ASB the greatest. In the plane ASB make the angle $BSD = BSC$, draw any line ADB; and having taken $SC = SD$, join AC, BC. Then AS and SC, are equal to AS and SD; but $BC = BD$ by the equal triangles BSC, BSD; $\therefore AD < AC$, $\therefore ASD < ASC$, $\therefore ASC + CSD > ASB$.

A sphere is a solid terminated by a curved surface, of which all the points are equally distant from an interior point, which we call the centre.

We may conceive a sphere generated by the rotation of a semi-circle around one of its diameters, for the surface so generated will have all its points equidistant from the centre.

Every section of a sphere is a circle; "for conceive three lines from the centre to points in the section, they are equal, as being radii of the sphere; from the centre conceive a perpendicular on the plane of the section, taking the square of this from the square of each of the lines already drawn, the squares of the lines in the plane of the section from the foot of the perpendicular are equal, $\therefore$ these lines are themselves equal, and $\therefore$ this section a circle." We call a *great circle* that section the plane of which passes through the centre of the sphere, and all others are called *small circles.*

A *plane* is a *tangent* to a sphere, when it has but one point in common with its surface.

The *pole of a circle* of a sphere is a point equally distant from every point in the circumference of this circle.

Two great circles always bisect each other, for their common intersection passing through the centre is a diameter.

Every great circle divides the sphere and its surface into two equal parts.

Two points on the surface of a sphere determine the plane of a great circle, for the two given points and the centre of the sphere are three points that determine the position of a plane. The only exception to this is when the points are diametrically opposite.

Every plane perpendicular to the radius at the surface of the sphere is a tangent plane; "for conceive any point in this plane different from that where it meets the radius in question, this point is distant from the centre of the sphere by the hypothenuse of a right angled triangle, one of the sides of which is the radius."

Two spheres mutually touch when the distance of their centres is equal to the sum or difference of their radii.

If a diameter be drawn perpendicular to the plane of a great circle, it willl intersect the surface of the sphere in points that are poles to this circle, and to every small circle parallel to it; "for the line oo' (Fig. 7.) being perpendicular to the plane of the circle nrq, is perpendicular to all the right lines mv, mr, that meet it in that plane, $\therefore$ the arcs ov, or, are equal, and each 90°, $\therefore$ o is the pole of nrq. Again, oo' being perpendicular to the plane of the circle $n'r'q'$, passes through its centre by what has been before proved, $\therefore m'v' = m'r'$, $\therefore$ the chord $ov' = or'$, and consequently the arcs ov' or' are equal, whence o is the pole of the small circle $n'r'q'$".

Every great circle passing through the poles of another great circle is at right angles to it; "for the plane of the former is at right angles to the plane of the latter, since it passes through a line perpendicular thereto; hence the arcs cut at right angles, since the angles they make with each other is the same as that of the planes."

Hence to find the pole of a great circle, draw a great circle at right angles to it, and take on the latter an arc equal to 90°; or, draw two great circles at right angles to it, and their intersection will be the pole.

A portion of a great circle, rv, is to the corresponding portion of a parallel to it, $r'v'$, as the radius of the great circle mr to the radius of the small one $m'r'$. But $m'r'$ is the *sine* of the arc or', mr being the radius, $\therefore$ arc vr : arc $v'r'$ :: 1 : sin or' :: 1 : cos rr'.

The spherical figure, $ovo'r$, contained by two arcs of great circles, is called a *lune;* and the angle of the lune, rov, is measured by the arc rv, of the great circle to which the sides of the lune are at right angles.

The area of the *semi-lune rov* : area of the hemisphere :: angle of the lune : 360°; $\therefore$ let A be the number of degrees in the angle of the lune, and the area of the lune $= S.\ \frac{A^\circ}{360^\circ}$ where S denotes the surface of the sphere.

The area of the surface of the sphere is four times that of its

great circle, or $S=4\pi r^2$.* Hence the area of the lune $= 4\pi r^2 . \frac{A^\circ}{360^\circ} = 2r^2 . A^\circ$. To the radius unity the area of the lune $= 2A^\circ$.

A spherical triangle is a portion of the surface of a sphere contained by three arcs of great circles.

All angles of a spherical triangle are to be understood to be under 180°, should an angle B, become 180°, as in (Fig. 8.), the triangle becomes the lune ABC'; should it become greater, the triangle ADC'' will answer in its place; so ∴ the consideration of any triangle that has an angle > 180° is useless.

A spherical triangle is *Isosceles*, *right* angled, *equilateral*, &c. in the same cases as a rectilinear triangle.

Any two sides of a spherical triangle are together less than the third side; "for the sides are the measures of the solid angles at the centre of the sphere, any two of which are greater than the third, as has been proved already."

Hence the sum of the three sides of a spherical triangle is $< 360^\circ$; "for if any two sides, AB, AC, (Fig. 9.) be produced to meet in D, the periphery of the lune $AD=360^\circ$; but this is greater than the periphery of the triangle for $BD+DC>BC$."

In like manner it might be shewn that the sum of the sides of any spherical polygon is $< 360^\circ$. Let us take, for instance, the pentagon $ABCDE$, (Fig. 10.); it is converted into a quadrilateral by producing the sides AB, DC, to meet in F, and we have the perimeter of the quadrilateral > that of the pentagon, and this again, by producing the sides, AE, DC, to meet in G, is con-

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* This may be proved thus: By the principles of the differential calculus, $dS=2\pi y . \sqrt{dx^2+dy^2}$; but $\sqrt{dx^2+dy^2} = \frac{rdy}{\sqrt{r^2-y^2}}$; $\therefore dS = \frac{2\pi r . ydy}{\sqrt{r^2-y^2}}$, $\therefore S=2\pi r\sqrt{r^2-y^2}+c$. This taken between the limits $y=o$ and $y=r$ would give $2\pi r^2$ for the area of the hemisphere, $\therefore S=4\pi r^2$. Similarly, the surface of an ellipsoid of revolution is four times that of the ellipse passing through its poles.

verted into a triangle, the perimeter of which, we have also still $>$ that of the quadrilateral, and $\therefore$ than that of the pentagon; but the perimeter of the triangle is $< 360°$, $\therefore$ *a fortiori*, that of the pentagon is $< 360°$

The same proof would apply to any other polygon. This is, at bottom, the same as the proposition by which it is proved that the several angles that constitute the faces of a solid angle are together $<$ 4 right angles. Neither is necessarily true, if the solid contain any re-entrant *edges;* or, in other words, if any side of the spherical polygon produced cut the figure.

From what has been said of the area of the lune, we can find very simply the area of a spherical triangle. " In (Fig. 11.) let ABC be a spherical triangle; complete the side AB into an entire circle, and produce the sides BC, AC, to meet in C', the triangle $A'B'C'$, is obviously the same as ABC, then observing the three lunes, AA', BB', CC', it is obvious that the hemisphere falls short of the sum of these three lunes by twice the area, which we shall call Σ; $\therefore$ $\Sigma = r^2(A+B+C) - \pi r^2 = r^2 \left\{(A+B+C) - 180°\right\}$.
Hence to radius unity, $\Sigma = (A + B + C) - 180°$.

This very simple solution is taken from the works of Wallis, but the Theorem is due to Albert Girard. The result shews us that the area of a spherical triangle is proportional to the excess of the sum of its three angles above 180°.

The quantity Σ cannot certainly be greater than half the surface of the globe, since no side can be greater than 180°. If Σ were half the surface of the globe, then $\Sigma = 2\pi r^2 = r^2\,(540° - 180°)$. And the minor limit of the area is cypher, or $\Sigma = r^2\,(180° - 180°)$.

Hence it appears simply and clearly that the sum of the three angles of a spherical triangle is $> 180°$ and $< 540°$.

The following method of mechanically determining the area is due to De Gua (*Mem. Acad. des Sc.* 1783). Describe a great circle of the sphere *in plano*, measure around its circumference an arc equal to the sum of the three angles of the triangle, draw from the extremities of this arc two diameters, these diameters will resolve the circle into four sectors, the sum of the two small sectors is the area, if the sum of the three angles be $> 180°$ and $< 270°$; and the sum of the two large ones, if $> 270°$ and $< 360°$. If the sum of the angles be $> 360°$, it will be necessary to describe two

great circles, and to measure on each an arc equal to the semi-sum of the three angles.

One circle might be made to serve if we measure on it an arc equal to the excess of its angles above $360°$, and find the sum of the opposite sectors. This, added to the surface of a great circle, gives the area.

From what has been said of the area of the triangle, it may easily be seen that the surface of any spherical polygon of n sides is proportional to the sum of its angles, *minus* $(n-2).\ 180°$; or

$$\Sigma = \left\{(A+B+C+\&c.)-(n-2)\ 180°.\right\}\ r^2.$$

It might hence be deduced, as it has been for the triangle, that the sum of the angles of any spherical polygon is $> (n-2).\ 180°$ and $< n.\ 180°$.

Having thus briefly explained the nature of the sphere, and the true meaning of spherical triangles, we shall proceed to the deduction of one fundamental formula, from which, by the aid of analytic reasoning, we shall deduce all others. Spherical Trigonometry, or the Theory of Spherical Triangles, is but an application of the general theory of sines, and an application of it perfectly analogous to the resolution of plane triangles given in Chap. III. of this work. From this analogy in their nature it is desirable that both these applications be deduced from an analogous fundamental formula. In the present application, however, the deduction of such a formula is not altogether so optional as in the former. Euler, in a memoir the most elegant of any up to his time, entitled *Trigonometria universa ex primis principiis derivata*, and which the reader will find in the Petersburgh Acts for 1779, has founded the whole of Spherical Trigonometry upon three equations. Subsequently to him De Gua had conceived the idea of making all depend upon a single property, but his memoir on the subject is filled with calculations so complicated, that they seem adapted more to shew the inconvenience of the method than to cause it to be adopted. A supply to this deficiency, however, was soon afforded by the celebrated La Grange.* He, as well as the subsequent improvers of this science, Delambre and Puissant, have commenced with the same fundamental formula as De Gua, but

* Journal de l'Ecole Polytechnique, Vol. II. page 270.

used in their subsequent proceeedings very different methods. Our subject at present is to endeavour, from amongst such scattered and diversified materials, to form a clear and sufficient elementary system of formulæ, and to apply them to the resolution of the various cases of spherical triangles.

We shall begin with expressing the cosine of an angle of a spherical triangle in terms of the sines and cosines of its sides. Let A, B, C, (Fig. 12.) be three points, on the surface of the sphere whose centre is o, connected by arcs of great circles, a, b, c; draw the radii oA, oB, oC, and draw tangents to the arcs b, c, at the point A, to meet the produced radii oB, oC, at the points p, m. Then $pm^2 = pA^2 + Am^2 - 2\,pA.\,Am.\cos A$; and again, $pm^2 = po^2 + om^2 - 2\,po.\,om.\cos pom$. We have $\therefore$ from these two equations, using Trigonometrical language, $\tan^2 b + \tan^2 c - 2.\tan b.\tan c.\cos A = \sec^2 b + \sec^2 c - 2.\sec b.\sec c.\cos a$. Subtract from both sides of this last equation $\tan^2 b + \tan^2 c$, divide all the terms by 2, and multiply all by $\cos b.\cos c.$ and the result obtained is $\cos a = \cos A.\sin b.\sin c + \cos b.\cos c.$ (a)

$$\therefore\ \cos A = \frac{\cos a - \cos b \cos c}{\sin b.\sin c} \qquad (1)$$

This is a formula perfectly analogous to that used at the commencement of Chap. III.

The process for finding $\cos B$ and $\cos C$ would be just the same; $\therefore$ we have

$$\cos B = \frac{\cos b - \cos a.\cos c}{\sin a.\sin c}$$

$$\cos C = \frac{\cos c - \cos a.\cos b}{\sin a.\sin b}$$

Formula (a) admits of various interesting simplifications. For instance, if $b = c$, the formula becomes $\sin \tfrac{1}{2}a = \sin\tfrac{1}{2}A \sin b$ (b). This completely resolves the Isosceles triangle.

Drawing the bisector, β, of the vertical angle of an Isosceles triangle, let s, s' be the segments into which it divides the base, then $\cos s = \cos \tfrac{1}{2}A.\sin b.\sin \beta + \cos b.\cos \beta$, and $\cos s' = \cos \tfrac{1}{2}A.\sin c.\sin \beta + \cos c.\cos \beta$, but $b = c$, $\therefore s = s'$, or, *the bisector of the vertical angle also bisects the base in an Isosceles spherical triangle.*

Again, let B', C', be the angles this bisector makes with the base, then $\cos b = \cos B'.\cos\beta.\cos\frac{1}{2}a + \sin\beta.\sin\frac{1}{2}a$, and $\cos c = \cos C'.\cos\beta.\cos\frac{1}{2}a + \sin\beta.\sin\frac{1}{2}a$, $\therefore$ since $b=c$ we have $B'=C'=90°$, or, *the bisector of the vertical angle is at right angles to the base, and, $\therefore$ e converso, the perpendicular bisector of the base passes through the vertical angle.*

Again, $\cos\beta = \cos B \sin\frac{a}{2}.\sin c + \cos\frac{a}{2}.\cos c$, and $\cos\beta = \cos C.\sin\frac{a}{2}.\sin b + \cos\frac{a}{2}.\cos c$; but since $b=c$, $\cos B = \cos C$, $\therefore B=C$; or the angles at the base of an Isosceles spherical triangle are equal.

By formula (b), and what has been proved, it appears that the sine of the side of a right angled triangle is equal to the sine of the hypothenuse multiplied by the sine of the opposite angle, hence in any spherical triangle, from the angle A draw a perpendicular arc, β, on the side a; then $\sin\beta = \sin B.\sin c$, and also $\sin\beta = \sin C.\sin.b$; $\therefore \frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}$. (2)

Also, we have $\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}$, $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$.

We can now pass directly to an expression for the sine of an angle in terms of the sines of the sides, we have

$$\sin^2 B = \frac{\sin^2 b}{\sin^2 c}\left\{1 - \frac{(\cos c - \cos a.\cos b)^2}{\sin^2 a.\sin^2 b}\right\} =$$

$$\frac{2\cos a.\cos b.\cos c - \cos^2 c - (\cos^2 a.\cos^2 b - \sin^2 a.\sin^2 b)}{\sin^2 a.\sin^2 c};$$ but $2\cos a.\cos b = \cos(a+b) + \cos(a-b)$, and $\cos^2 a.\cos^2 b - \sin^2 a.\sin^2 b = \cos(a+b).\cos(a-b)$ $\therefore$ making these substitutions, and decomposing into factors, we have

$$\sin^2 B = \frac{\left\{\cos c - \cos(a+b)\right\}.\left\{\cos(a-b) - \cos c\right\}}{\sin^2 a.\sin^2 c},$$

using the formula for the difference of the cosines of two arcs, we have (taking the square root)

$$\sin B = \frac{2}{\sin a.\ \sin c}.\ \sqrt{\sin s.\ \sin(s-a).\ \sin(s-b).\ \sin(s-c)} \quad (3)^*$$

Similarly for A and C we have

$$\sin A = \frac{2}{\sin b.\ \sin c}.\ \sqrt{\sin s.\ \sin(s-a).\ \sin(s-b).\ \sin(s-c)}$$

$$\sin C = \frac{2}{\sin a.\ \sin b}.\ \sqrt{\sin s.\ \sin(s-a).\ \sin(s-b).\ \sin(s-c)}$$

If we add together the values of $\cos A$ and $\cos B$, and substitute in them for $\cos c$, from equation (a), we find, after reduction, a formula sometimes of use, viz. $\cos A + \cos B = \dfrac{2.\ \sin(a+b).\ \sin^2\frac{C}{2}}{\sin c}$ (4)

By (a) $\cos a = \cos A.\ \sin b.\ \sin c + \cos b.\ \cos c$, substituting in this for $\sin c$ and $\cos c$, dividing both sides by $\sin a$, arranging, and then dividing both sides by $\sin b$, we have ultimately

$$\cot A.\ \sin C = \cot a.\ \sin c - \cos C.\ \cos b. \quad (5)$$

It is requisite to be in possession of formulæ for the cosine and sine of a side, in terms of the cosines and sines of the angles. This is obtained as follows; we have

$$\cos a = \cos b.\ \cos c + \sin b.\ \sin c.\ \cos A,$$

$$\cos c = \cos a.\ \cos b + \sin a.\ \sin b.\ \cos C,$$

Substituting from the second of these equations for $\cos c$ in the first, it becomes $\cos a = \cos a.\ \cos^2 b + \sin a.\ \sin b.\ \cos b.\ \cos C + \sin b.\ \sin c.\ \cos A$; and replacing $(1-\cos^2 b)$ by $\sin^2 b$, and dividing by $\sin b$, we have $\sin b.\ \cos a = \sin a.\ \cos b.\ \cos C + \sin c.\ \cos A$. Similarly we have $\sin a.\ \cos b = \cos a.\ \sin b.\ \cos C + \sin c.\ \cos B$. In the former of these, substituting from the latter for $\sin a.\ \cos b$; we have $\cos a.\ \sin b = \cos a.\ \sin b.\ \cos^2 C + \sin c.\ \cos B.\ \cos C + \sin c.\ \cos A$; that is to say, $\sin b.\ \sin^2 C = \sin c \left\{\cos B.\ \cos C + \cos A\right\}$

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* S denoting here, as in Plane Trigonometry, the semi-sum of the three sides.

substituting for sin c its value, sin b. $\frac{\sin C}{\sin B}$ and dividing by sin b. sin C, we have, arranging the terms, $\cos a = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C}$. (6)

This formula, it appears, is in every respect similar to the formula for cos A in terms of the sides, the angles having taken place of the sides, and *vice versâ*; also, the cosines have become negative, the sines remaining positive.

Let us conceive a triangle, the three angles of which are A', B', C', and the sides a', b', c'; in this triangle we have $\cos A' = \frac{\cos a' - \cos b' . \cos c'}{\sin b' . \sin c'}$; now should the sides of this triangle be supplemental to the angles of the original one, and the angles of this to the sides of the original, this formula would be converted into formula (6) by substituting for A', $(180 - A)$; for b', $180 - b$, &c.

This triangle is called the *supplemental* triangle, from its property just alluded to, and by the French mathematicians the *polar* triangle, from the mode of its description, which is as follows. With the angles, A, B, C, of the given triangle as poles, (Fig. 13.) describe arcs of great circles to form the triangle whose angles are A', B', C', then it is obvious, from the figure, that the angle A and side $C'B'$ are supplemental, as also the angle A' and the side a; the same is obviously true of the other sides and angles.

The consideration of this triangle is useful in reverting some of the propositions that we have already proved; for instance, to prove that if the angles at the base of a triangle be equal, the sides are also equal. Let us take the polar triangle, then the angles of the given triangle being equal, the sides of the polar triangle are equal, ∴ the angles of the polar, ∴ the sides of the given triangle, being supplemental to these latter.

It may also be proved by the polar triangle, that the three angles of a spherical triangle are $> 180°$, and $< 540°$ for $A+B+C+a'+b'+c' = 540°$; but $a'+b'+c' < 360$, ∴ $A+B+C > 180°$. Again, $a'+b'+c' > 0$, ∴ $A+B+C < 540$.

To form an expression for sin a, we have in the polar triangle $\sin A' = \frac{2}{\sin b' . \sin c'} \sqrt{\sin s . \sin (s-a') . \sin (s-b') \sin (s-c')}$, as

appears from formula (3) of this chapter. Here $S = \frac{a'+b'+c'}{2} = 270 - \frac{A+B+C}{2}$, $\therefore \sin s = -\cos \frac{A+B+C}{2} = -\cos S'$, using S' to denote the semi-sum of the three angles; and $s - a' = 90 - \left(\frac{A+B+C}{2} - A\right)$, $\therefore \sin (s - a') = \cos \left(\frac{A+B+C}{2} - A\right) = \cos (S' - A)$, changing in a similar manner $(s - b')$ and $(s - c')$ we finally obtain

$$\sin a = \frac{2}{\sin B . \sin C} \sqrt{\sin S' . \sin (S' - A) . \sin (S' - B) . \sin (S' - C)} \quad (7)$$

Having thus laid a foundation for the deduction of formulæ, we shall now proceed to apply them to their proper use in the solution of triangles, both right angled and oblique.

CHAP. V.

ON THE RESOLUTION OF RIGHT ANGLED TRIANGLES—CIRCULAR PARTS—NAPIER'S RULES—QUADRANTAL TRIANGLES, &c.

In the preceding chapter we have obtained the four following formulæ,

$$\cos c = \cos a . \cos b + \sin a . \sin b . \cos C,$$

$$\sin b . \sin C = \sin c . \sin B,$$

$$\cos a = \cos c . \cos b + \sin c . \sin b . \cos A,$$

$$\cos c = \frac{\cos C + \cos B . \cos A}{\sin B . \sin A}.$$

In order to adapt these to the particular case of right angled triangles, let us suppose C a right angle. The formulæ then will be readily observed to become the following.

1. $\cos c = \cos a . \cos b$, or, $\sin (90^\circ - c) = \cos a . \cos b$.

2. $\sin b = \sin c . \sin B$, or, $\sin b = \cos (90^\circ - c) . \cos (90^\circ - B)$.

3. $\cos A = \cot c . \tan b$, or, $\sin (90^\circ - A) = \tan (90^\circ - c) . \tan b$.

4. $\cos c = \cot A . \cot B$, or, $\sin (90^\circ - c) = \tan (90^\circ - A) . \tan (90^\circ - B)$.

From these we may easily derive the six following:

5. $\cos B = \cot c . \tan a$, or, $\sin (90^\circ - B) = \tan (90^\circ - c) . \tan a$.

6. $\sin a = \sin c . \sin A$, or, $\sin a = \cos (90^\circ - c) . \cos 90^\circ - A)$.

7. $\cot A = \cot a . \sin b$, or, $\sin b = \tan (90^\circ - A) . \tan a$.

8. $\cot B = \cot b . \sin a$, or, $\sin a = \tan (90^\circ - B) \tan b$.

9. $\cos A = \cos a . \sin B$, or, $\sin (90^\circ - A) = \cos a . \cos (90^\circ - B)$.

10. $\cos B = \cos b . \sin A$, or, $\sin (90^\circ - B) = \cos b . \cos (90^\circ - A)$.

These ten formulæ resolve all the cases of right angled spherical triangles.

They are capable of being all comprised under two very simple enunciations. Let us consider the triangle made up of five parts; the two sides, the complement of the hypothenuse, and complements of the base angles; then calling any part *a middle*, the two parts next it in the triangle *adjacents*, and the two remaining parts separated, each from the middle, by an adjacent part, *opposites*; it is universally true, and it is sufficient, as appears from the above formulæ, to say,

1° The rectangle of the sine of middle and radius = rectangle of tangents of parts adjacent.

2° The rectangle of the sine of middle and radius = rectangle of cosines of opposite parts.

These are the rules of Napier, which he has given in a very obscure manner at the end of his work, entitled, '*Mirifici Canonis Constructio.*' *M. Mauduit*, in his Trigonometry, has presented these rules in a different, and, perhaps, a more commodious form;

he considered the parts to be the complements of the sides, the base angles, and the hypothenuse. The rules would then be as follow:

1° The rectangle of cos M and rad. = product of cotangents of parts adjacent.

2° The rectangle of cos M and rad. = product of sines of parts opposite.

For comprehensiveness, and facility of being remembered, these rules of Napier have been acknowledged to stand unrivalled. Delambre, however, objects to them: 1° From the trouble of using the complements of some of the quantities; 2° From the trouble of settling the middle; 3° Because the quantity sought is sometimes amongst the middles, and sometimes amongst the extremes.—*Astronomie, tom.* 1, p. 205.

Should two angles of the triangle become right, the two sides opposite them become quadrants, and the third side equal to the third angle.

Should the side c become a quadrant, the triangle becomes what is called the quadrantal triangle, and rules may be given for it similar to Napier's, using as parts the complements of the sides, the complement of the vertical angle, and the base angles. In this case, however, we can always use the right angled triangle, whose sides are, one side of the given one, the angle opposite said side in the given one, and the complement of the other side.

E. g.—Given the sun's polar distance and complement of the latitude, to find the angle from noon at the time of rising. The triangle is quadrantal, but we may use the triangle whose sides are the lat. polar dist. and azimuth, and find the angle from midnight.

Woodhouse advises to use the polar or supplemental triangle, which would be far more inconvenient and awkward than solving even the given quadrantal triangle by almost any method.

We shall proceed to give some numerical illustration of the resolution of right angled triangles.

Example 1.—Given the polar distance of a celestial object,

$c=51° 4' 34''.6443$, and the hour angle from the meridian, $A=30°$, to find the zenith distance, a, of the object when on the prime vertical.

We have $r.\sin a=\sin c.\sin A, \therefore \log.\sin a=\log.\sin c+\log.\sin A-10.$

$$\begin{array}{lcl} \log.\sin c & = & 9.8909702298 \\ \log.\sin A & = & 9.6989700043 \\ -\log. r & = & -10 \\ \hline \text{Sum,} & = & 9.5899402341 \end{array}$$

Hence we obtain $a=22° 53' 30''.3679$.

If in this case we had been given a and A to find c, b, B, the cases would be ambiguous, for they would be found by their sines, and $\therefore$ may belong to the triangle formed by producing c and b to meet at the south pole, for the same data exist in this triangle. And, in general, if the data be an angle, and the side opposite, the three formulæ for the other parts become ambiguous.

Example 2.—In the same case to compute b, from knowing c and a as above. We have $r.\cos c=\cos a.\cos b, \therefore \log.\cos b= 10+\log.\cos c-\log.\cos a$.

$$\begin{array}{lcl} \log.\cos c & = & 9.7981567280 \\ \log. r & = & 10 \\ -\log.\cos a & = & -9.9643733977 \\ \hline \log.\cos b & = & 9.8337833303 \end{array}$$

$\therefore b=47° 0' 0''$.

Example 3.—Given the polar distance of an object when rising, $c=51° 4' 34''.6443$, and the latitude of the place, $a=22° 53' 30''.3679$, to find the time of rising, or the angle B. We have $r.\cos B= \tan a.\cot c, \therefore \log.\cos B=10+\log.\tan a-\log.\tan c$.

$$\begin{array}{lcl} \log.\tan a & = & 9.6255668364 \\ \log. r & = & 10 \\ -\log.\tan c & = & -10.0928135018 \\ \hline \text{Sum,} & = & 9.5327533346 \end{array}$$

$\therefore B=70° 3' 44''.0601$.

There are cases of right angled triangles apparently ambiguous; for instance, from the hypothenuse and one angle, B, to find the side opposite, b, we have $\sin b = \sin c . \sin B$; but it is apparently doubtful whether we ought to use b or $(180 - b)$. The ambiguity is removed by remarking, that if b is $>$ or $< 90°$; B is $>$ or < 90.

We will now proceed to investigate and apply formulæ for the solution of oblique angled triangles.

CHAP. VI.

ON THE RESOLUTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

Before proceeding to a distinct enumeration of the cases, let us deduce other formulæ in addition to the four fundamental ones given in Chap. IV.

We have already had the following equations:

$$\cos A = \frac{\cos a - \cos b . \cos c}{\sin b . \sin c.}$$

$$\cos B = \frac{\cos b - \cos a . \cos c}{\sin a . \sin c} \qquad (A)$$

$$\cos C = \frac{\cos c - \cos a . \cos b}{\sin a . \sin b}$$

These formulæ not being adapted to logarithmic computation, are inconvenient for finding the angles from knowing the sides. They require, previously to being of such use, some transformations.

The six following formulæ are instantly deducible from them, by simple addition and subtraction, and by the formulæ for the cosine of the sum and difference of two arcs.

$$1 - \cos A = \frac{\cos(b-c) - \cos a}{\sin b.\ \sin c}$$

$$1 - \cos B = \frac{\cos(a-c) - \cos b}{\sin a.\ \sin c}$$

$$1 - \cos C = \frac{\cos(a-b) - \cos c}{\sin b.\ \sin a}$$

$$1 + \cos A = \frac{\cos a - \cos(b+c)}{\sin b.\ \sin c} \qquad (B)$$

$$1 + \cos B = \frac{\cos b - \cos(a+c)}{\sin a.\ \sin c}$$

$$1 + \cos C = \frac{\cos c - \cos(a+b)}{\sin a.\ \sin b}$$

By formula (20) Chap. II. the difference of the cosines of two arcs $=$ 2. sin $\frac{1}{2}$ sum $\times$ sin $\frac{1}{2}$ diff. $\therefore$ $\cos(b-c) - \cos a =$ $2.\sin\frac{a+b-c}{2}.\sin\frac{a+c-b}{2}$; or, $\left(\text{if } S = \frac{a+b+c}{2}\right)$, $= 2.\sin(S-c).$ $\sin(S-b)$. Again, $1-\cos A = 2\sin^2\frac{A}{2}$, and $1+\cos A = 2\cos^2\frac{A}{2}$. Introducing these changes on the formulæ of class (B) we obtain the six following formulæ:

$$\sin^2\frac{A}{2} = \frac{\sin(s-c).\ \sin(s-b)}{\sin b.\ \sin c}$$

$$\sin^2\frac{B}{2} = \frac{\sin(s-a).\ \sin(s-c)}{\sin a.\ \sin c}$$

$$\sin^2\frac{C}{2} = \frac{\sin(s-a).\ \sin(s-b)}{\sin b.\ \sin a}$$

$$\cos^2\frac{A}{2} = \frac{\sin s.\ \sin(s-a)}{\sin b.\ \sin c} \qquad (C)$$

$$\cos^2\frac{B}{2} = \frac{\sin s.\ \sin(s-b)}{\sin a.\ \sin c}$$

$$\cos^2\frac{C}{2} = \frac{\sin s.\ \sin(s-c)}{\sin a.\ \sin b}$$

In class (C), dividing the first formula by the fourth, the second by the fifth, and the third by the sixth, we obtain,

$$\tan^2 \tfrac{1}{2}A = \frac{\sin (s-c). \sin (s-b)}{\sin s. \sin (s-a)}$$

$$\tan^2 \tfrac{1}{2}B = \frac{\sin (s-c). \sin (s-a)}{\sin s. \sin (s-b)} \qquad (D)$$

$$\tan^2 \tfrac{1}{2}C = \frac{\sin (s-a). \sin (s-b)}{\sin s. \sin (s-c)}$$

These formulæ afford us means of computing the angles of a triangle in terms of its sides, by means of the sines, cosines, or tangents of their halves.

Multiplying the first and fifth of the class (C), we have $\sin \frac{A}{2}.\cos \frac{B}{2} = \frac{\sin (s-b)}{\sin c} . \sqrt{\frac{\sin s. \sin (s-c)}{\sin b. \sin c.}}$; and from the second and fourth of the same class, we have $\cos \frac{A}{2}. \sin \frac{B}{2} = \frac{\sin (s-a)}{\sin c}.\sqrt{\frac{\sin s. \sin (s-c)}{\sin b. \sin c}}$. Adding these expressions, substituting for the left hand member of the equation, its value, $\sin \frac{1}{2}(A+B)$; for the radical, its value, $\cos \frac{C}{2}$; and for $\frac{\sin (s-a) + \sin (s-b)}{\sin c}$ its value, $\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c}$; we obtain

$$\sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2} C.\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2} c}$$

Similarly, $$\sin \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2} C\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2} c}$$

$$\cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} C\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2} c} \qquad (E)$$

$$\cos \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2} C\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2} c}$$

These remarkable and elegant formulæ are due to Delambre, who demonstrates them by a much more circuitous process.—*Vide Astronomie, tom.* 1, *chap.* 10. They are here deduced very nearly as M. Puissant deduces them in his *Traitè de Geodesie.*

Of the formulæ in class (E), dividing the first by the third, and the second by the fourth; and again, dividing the fourth by the third, and the second by the first, we obtain

$$
\left.
\begin{aligned}
\tan \tfrac{1}{2}(A+B) &= \cot \tfrac{1}{2}C.\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \\
\tan \tfrac{1}{2}(A-B) &= \cot \tfrac{1}{2}C.\,\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} \\
\tan \tfrac{1}{2}(a+b) &= \tan \tfrac{1}{2}c.\,\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \\
\tan \tfrac{1}{2}(a-b) &= \tan \tfrac{1}{2}c.\,\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}
\end{aligned}
\right\} \quad (F)
$$

Thus we have arrived, certainly in the most direct and natural manner, at the four analogies of Napier, as the four formulæ of class (F) are designated. They completely solve the two cases of spherical triangles, in which we know two sides and the included angle, or two angles and the included side.

We have from the first and second of formulæ (F), $\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \dfrac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)}$, a formula analogous to that for solving the similar case in plane Trigonometry. For spherical triangles this formula would be insufficient, for ($A+B$) and ($A-B$) are equally unknown from knowing the third angle C.

If our object were to find one angle alone, it may be done by formula (5), (Chap. 4.) already demonstrated.

From equations in class (C) we may deduce the following relations.

$$2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=\sin A=\frac{2}{\sin b.\sin c}\cdot\sqrt{\sin s.\sin(s-a).\sin(s-b).\sin(s-c)}$$

$$\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C=\frac{\sin(s-a).\sin(s-b).\sin(s-c)}{\sin a.\sin b.\sin c}$$

$$\tan\tfrac{1}{2}A.\tan\tfrac{1}{2}B.\tan\tfrac{1}{2}C=\frac{\sqrt{\sin(s-a).\sin(s-b).\sin(s-c)}}{\sqrt{\sin s.\sin s.\sin s}}.$$

The first of these formulæ has been already arrived at by a much less elegant process. The other two are remarkable for their simplicity, if we consider that they involve every part of the spherical triangle.

Recollecting also that $\cos\tfrac{1}{2}(A+B+C)=\cos\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C-\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\cos\tfrac{1}{2}C-\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}C.\cos\tfrac{1}{2}B-\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C.\cos\tfrac{1}{2}A$, we may easily collect from the formulæ in class (C) that

$$\cos\tfrac{1}{2}(A+B+C)=-\frac{\sqrt{\sin s.\sin(s-a).\sin(s-b).\sin(s-c)}}{2.\cos\tfrac{1}{2}a.\cos\tfrac{1}{2}b.\cos\tfrac{1}{2}c.}$$

(*Vide Cagnoli*, p. 329).

Since $\cos\tfrac{1}{2}(A+B+C)$ is essentially negative, $\tfrac{1}{2}(A+B+C)>90°$ and $<270°$ $\therefore$ *the sum of the three angles of a spherical triangle is greater than two right angles, and less than six*, which is a theorem that has been already proved in two different ways.

After the same manner that we have obtained an expression for the sine of a side in terms of the sines of angles, by means of the polar triangle, we may obtain expressions for the sine and cosine of half a side. In the polar triangle we have $\cos\tfrac{1}{2}A'=\sqrt{\frac{\sin s.\sin(s-a')}{\sin b'.\sin c'}}$, or since $\cos\tfrac{1}{2}A'=\cos\frac{(180°-a)}{2}=\sin\frac{a}{2}$, we have

$$\sin\frac{a}{2}=\sqrt{\frac{-\cos S'.\cos(S'-A)}{\sin B.\sin C}}$$

Similarly, $$\cos\frac{a}{2} = \sqrt{\frac{\cos(S'-B).\cos(S'-C)}{\sin B.\sin C}}$$

and $$\tan\frac{a}{2} = \sqrt{\frac{-\cos S'.\cos(S'-A)}{\cos(S'-B).\cos(S'-C)}}$$

Having thus prepared a clear and sufficient system of formulæ, our next business shall be to apply them to their proper use in the solution of the different cases of spherical oblique angled triangles. For this purpose it is not merely necessary to be furnished with equations exhibiting the possibility of such solution, it is necessary to be furnished with formulæ from which the quantities required may be immediately and directly deduced by logarithmic computation. Were the former mode of solution sufficient, we might have ended with the preceding chapter, as therein is contained every thing requisite for a merely theoretical solution of any case that could present itself.

In an oblique angled spherical triangle there are six quantities, from three of which being given, a fourth may be found. We must then have equations between four of those quantities, combined in all possible ways. Of such combinations there are fifteen, which may be classed as follows:

$$(abcA),\ (abcB),\ (abcC),$$

$$(ABCa),\ (ABCb),\ (ABCc),$$

$$(aCbA),\ (aCbB),\ (aBcA),\ (aBcC),\ (bAcB),\ (bAcC),$$

$$(aAbB),\ (aCcA),\ (bCcB).$$

Amongst these there are to be found but six cases essentially distinct, which we shall accordingly proceed to enumerate.

First Case.—Given the three sides, to find an angle.

The following formulæ, which have been already proved, completely resolve this question.

$$1 \ldots\ldots \sin A = \frac{2}{\sin b.\ \sin c}.\sqrt{\sin s.\ \sin(s-a).\ \sin(s-b).\ \sin(s-c)}$$

$$2\ldots\ldots\cos\frac{A}{2}=\sqrt{\frac{\sin s.\ \sin(s-a)}{\sin b.\ \sin c}}$$

$$3\ldots\ldots\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b).\ \sin(s-c)}{\sin b.\ \sin c}}$$

$$4\ldots\ldots\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b).\ \sin(s-c)}{\sin s.\ \sin(s-a)}}$$

The value of $\cos A$ might be adapted to logarithmic computation, so as to afford a fifth formula. Let $\cos b.\ \cos c=\cos\theta$, where θ is an auxiliary arc, then $\cos A=\frac{\cos a-\cos\theta}{\sin b.\ \sin c}$, whence

$$5\ldots\ldots\cos A=\frac{2.\ \sin\frac{1}{2}(a+\theta).\ \sin\frac{1}{2}(a-\theta)}{\sin b.\ \sin c}$$

The subsidiary arc θ, in this last formula is $<$ or $>$ 90°, as the sides b, c, are of the same or different affection.

If all the angles be required, the first formula is convenient in point of brevity, as the radical in it once computed will equally serve for all. In point of accuracy, however, if one of the angles be nearly 90°, formula (1) is inconvenient, and any of the others might be used with advantage, especially (5), as an error on the cosine of such an arc will entail the least possible error on the arc. If the arc be nearly 180°, formula (4) is inconvenient, for then the variations on the tangent are not as the variations on the arc, whence the computation by proportional parts is not exact.

Of these formulæ, according to Delambre, the third should be most used, but the first is merely a matter of curiosity.

Second Case.—Given the three angles, to find a side.

The following formulæ solve this case.

$$1\ldots\sin a=\frac{2}{\sin B.\ \sin C}.\sqrt{-\cos S'.\cos(S'-A).\ \cos(S'-B).\cos(S'-C)}$$

$$2 \ldots \sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S'.\cos(S'-A)}{\sin B.\sin C}}$$

$$3 \ldots \cos \tfrac{1}{2} a = \sqrt{\frac{\cos(S'-B).\cos(S'-C)}{\sin B.\sin C}}$$

$$4 \ldots \tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S'.\cos(S'-A)}{\cos(S'-B).\cos(S'-C)}}$$

Since S' in these formulæ denotes the semi-sum of the three angles, it is obvious it must be some angle between 90° and 270°, and $\therefore \cos S' < 0$. And since, as has been proved in Chap. IV., $b'+c'>a'$ we have $180°-B+180°-C>180°-A \therefore B+C-A<180°$ $\therefore \frac{B+C-A}{2} < 90°$, but $S'-A=\frac{B+C-A}{2}$, $\therefore S'-A < 90 \therefore$ $\cos(S'-A) > 0$. For the same reason $\cos(S'-B) > 0$, $\cos(S'-C) > 0$.

This is a case that never occurs in practice.

Third Case.—Given two sides and the included angle.

By the analogies of Napier that have been proved, we have

$$\cos \tfrac{1}{2}(a+b) : \cos \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2} C : \tan \frac{A+B}{2}$$

$$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2} C : \tan \frac{A-B}{2}.$$

Thus we have the two angles A and B. To find the third side, c, we have $\sin c = \frac{\sin b}{\sin B}.\sin C$. However, as this is one of the cases that most frequently occurs in Astronomical calculations, it is requisite to be furnished with some separate and independant mode of determining c.

We have had already $\cos c = \cos a.\cos b + \sin a.\sin b.\cos C$. This expression, by substituting for $\cos C$ its value, 1—ver-sin C, by subtracting both sides of the equation from unity, and by writing

$2.\sin^2\frac{C}{2}$ for $1-\cos c$, becomes $2.\sin^2\frac{c}{2} = \text{ver-sin}\,(a-b) + \sin a.$ $\sin b.\ \text{ver-sin}\ C = \text{ver-sin}\ (a-b).\left(1 + \frac{\sin a.\sin b.\ \text{ver-sin}\ C}{\text{ver-sin}\ (a-b)}\right)$. Let us assume an auxiliary arc, θ, such that $\tan^2\theta = \frac{\sin a.\sin b.\ \text{ver-sin}\ C}{\text{ver-sin}\ (a-b)}$ (m). Then we have $2.\sin^2\frac{c}{2} = \text{ver-sin}\ (a-b).\ \sec^2\theta.$ (n). Converting (m) and (n) into logarithms, we have the two following equations for finding c.

$$2.\log.\tan\theta = \log.\sin a + \log.\sin b + \log.\text{ver-sin}\ C - \log.\text{ver-sin}\ (a-b) \quad (m')$$

$$2.\log.\sin\frac{c}{2} = \log.\text{ver-sin}\ (a-b) + 2\log.\sec\theta - \log.2 - 10. \quad (n')$$

Another formula may be arrived at by a different treatment of the value for $\cos c$. For $\cos C$, write $2.\cos^2\frac{C}{2} - 1$, subtract both sides of the equation from unity and there is had $2.\sin^2\frac{c}{2} = 1 - \cos\,(a+b) - 2.\sin a.\sin b.\cos^2\frac{C}{2} = 2.\sin^2\frac{a+b}{2} - 2.\sin a.$ $\sin b.\cos^2\frac{C}{2}$. Assume an arc, φ, such that $\sin^2\varphi = \sin a.\ \sin b.$ $\cos^2\frac{C}{2}$ then we have $\sin^2\frac{c}{2} = \sin^2\frac{a+b}{2} - \sin^2\varphi = \sin\left(\frac{a+b}{2} + \varphi\right).$ $\sin\left(\frac{a+b}{2} - \varphi\right) \therefore 2\log.\sin\frac{c}{2} = \log.\sin\left(\frac{a+b}{2} + \varphi\right) + \log.\sin\left(\frac{a+b}{2} - \varphi\right).$

This formula is used by Laplace to find the arc intercepted between two heliocentric positions of a comet, by means of the co-latitudes at these times, and the difference of longitudes.—*Mec. Cel. Livre* II. page 227.

There is still remaining another mode of obtaining the side c. From the formula for $\cos c$, by subtracting each side of the equation from unity, we obtain

$$\text{ver-sin}\ c = \text{ver-sin}\ (a-b) + \sin a.\ \sin b.\ \text{ver-sin}\ C.$$

This formula not being fitted for logarithmic computation would require a table of natural versed sines.

Fourth Case.—Given two angles and the side between, to find the remaining sides and the third angle.

By Napier's Analogies, already proved, we have

$$\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\cdot\tan\frac{c}{2}$$

$$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\cdot\tan\frac{c}{2}$$

Thus we have $\frac{a+b}{2}$ and $\frac{a-b}{2}$, whence $\therefore$ we have a and b.

It remains to find the third angle C. We have already proved the formula.

$$\cos C=\cos A.\cos B-\sin A.\sin B.\cos c.$$

$$=\cos A.\cos B-\sin A.\sin B.+2\sin A.\sin B.\sin^2\frac{c}{2}$$

$$=\cos(A+B)+2.\sin A.\sin B.\sin^2\frac{c}{2}$$

Subtracting both sides of this equation from unity, substituting, and dividing by 2, we obtain

$$\sin^2\frac{c}{2}=\sin\left(\frac{A+B}{2}+M'\right).\sin\left(\frac{A+B}{2}-M'\right)$$

where M' is such an arc that

$$\sin^2 M'=\sin A.\sin B.\sin^2\frac{c}{2}$$

These formulæ are the same as would be obtained from the 3d method in (Case 3), by the polar triangle. Others might be deduced from the other solutions of the same case.

Fifth Case.—Given two sides, a, b, and the angle A, to find B, C, and c.

$$\sin B = \frac{\sin A . \sin b}{\sin a}$$

$$\cot\frac{C}{2} = \tan\tfrac{1}{2}(A+B).\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}$$

$$\tan\frac{c}{2} = \tan\tfrac{1}{2}(a+b).\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}.$$

These three formulæ, which have been already proved, completely resolve this Case.

Sixth Case.—Given two angles, A, B, and a side, a, opposite one of them, to find b, c, and C.

$$\sin b = \sin a.\frac{\sin B}{\sin A}$$

$$\cot\frac{C}{2} = \tan\tfrac{1}{2}(A+B).\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}$$

$$\tan\frac{c}{2} = \tan\tfrac{1}{2}(a+b).\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}.$$

Having thus furnished formulæ for the six essentially distinct cases of spherical Trigonometry, we shall next proceed to give some numerical illustration of these cases calculated from the tables.

Example 1.—In case the first, let $a = 42° 8' 47''.9$; $b = 30°$; $c = 24° 3' 54''.6667$. By the second formula for this case we have

$$\log.\cos\frac{A}{2} = \tfrac{1}{2}\left\{20 + \log.\sin S + \log.\sin(S-a) - \log.\sin b - \log.\sin c\right\}$$

$S = 48° \ 6' \ 21''.29$ ....log. sin S = 9.8717949866
$S - a = 5° \ 57' \ 39''.36$ ......log. sin $(s-a)$ = 9.0162870555
2. log. r = 20.0000000000

Sum = 38.8880820421

log. sin b = 9.6989700043
log. sin c = 9.6104213585

Sum = 19.3093913628

$$\therefore \text{log. cos} \frac{A}{2} = 9.7893453396$$

$$\therefore A = 103° \ 59' \ 57''.5028.$$

In Vlacq's tables, the three last formulæ for the first Case are applied to the solution of this example; and indeed when only one angle is required, it would be an useless multiplication of labour to apply the first formula. If all the angles be required, the first formula can be as easily applied as any other.

Example 2.—In case the second, let $A = 103° \ 59' \ 57''.5$, $B = 46° \ 18' \ 7''.27$, and $C = 36° \ 7' \ 52''.44$. From the second formula we have

$$\text{log. sin} \frac{a}{2} = \tfrac{1}{2} \left\{ 20 + \text{log. cos} S + \text{log. cos}(S-A) - \text{log. sin} B - \text{log. sin} C \right\}$$

$S = 93° \ 12' \ 58''.35$ ....log. cos S = 8.7489934199
$S - A = -(10° \ 46' \ 59''.15)$ log. cos $(S-A)$ = 9.9922661960
2. log. r = 20.0000000000

Sum = 38.7412596159

log. sin B = 9.8591330480
log. sin C = 9.7705845247

Sum = 19.6297175727

$$\therefore \text{log. sin} \frac{a}{2} = 9.5557710216$$

$$\therefore a = 42° \ 8' \ 47''.9286.$$

This example is computed differently in the tables of Vlacq, from the polar triangle, and the result obtained precisely the same.

Example 3.—In case the third, let $a = 42° 8' 47''.9286$, $c = 24° 3' 54''.6667$, and $B = 46° 18' 7''.27$, to find A and C.

$$\log.\tan\frac{A+C}{2} = \log.\cot\frac{B}{2} + \log.\cos\frac{a-c}{2} - \log.\cos\frac{a+c}{2}$$

$$\frac{B}{2} = 23° 9' 3''.64 \ldots\ldots\ldots\ldots \log.\cot\frac{B}{2} = 10.3689736671$$

$$\frac{a-c}{2} = 9° 2' 26''.64 \ldots\ldots \log.\cos\frac{a-c}{2} = 9.9945709156$$

$$\text{Sum} = 20.3635445827$$

$$\frac{a+c}{2} = 33° 6' 21''.297 \ldots\ldots \log.\cos\frac{a+c}{2} = 9.9230689829$$

$$\therefore \log.\tan\frac{A+C}{2} = 10.4404755998$$

$$\therefore \frac{A+C}{2} = 70° 3' 54''.9753$$

$$\log.\tan\frac{A-C}{2} = \log.\cot\frac{B}{2} + \log.\sin\frac{a-c}{2} - \log.\sin\frac{a+c}{2}$$

$$\log.\cot\frac{B}{2} = 10.3689736671$$

$$\log.\sin\frac{a-c}{2} = 9.1962772372$$

$$\text{Sum} = 19.5652509043$$

$$\log.\sin\frac{a+c}{2} = 9.7373425161$$

$$\therefore \log.\tan\frac{A-C}{2}=9.8279083882$$

$$\therefore \frac{A-C}{2}=33^\circ\,56'\,2''.5275$$

$$\therefore A=103^\circ\,59'\,57''.5028;\; C=36^\circ\,7'\,52''.4478.$$

We shall here, as in the applications of the formulæ of Plane Trigonometry, propose some examples with their results, in order that the reader, if he find it necessary, may have in his power to exercise himself in such computations.

1° To compute the formulæ, $\sin A=\frac{N}{\sin b.\sin c}$;* $\sin B=\frac{N}{\sin a.\sin c}$; $\sin C=\frac{N}{\sin a.\sin b}$; when $a=50^\circ\,54'\,32''$; $b=37^\circ\,47'\,18''$; $c=74^\circ\,51'\,50''$.

$$R.\quad A=44^\circ\,10'\,40''$$

$$B=33^\circ\,22'\,45''$$

$$C=119^\circ\,55'\,6''$$

2° To compute the formula $\cos\frac{A}{2}=\sqrt{\frac{\cos s.\ \cos(s-b)}{\sin b.\ \sin c}}$ when $a=40^\circ$; $b=70^\circ$; $c=38^\circ\,30'$.

$$R.\quad A=31^\circ\,34'$$

3° To compute the formula $\sin^2\frac{c}{2}=\sin\left(\frac{a+b}{2}+M\right).\sin\left(\frac{a+b}{2}-M\right)$ when $a=70^\circ$; $b=38^\circ\,30'$; and the angle $C=31^\circ\,34'\,26''$.

$$R.\quad c=40^\circ\,0'\,0''.$$

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* $N=2.\sqrt{\sin s.\ \sin \overline{s-a}.\ \sin\overline{s-b}.\ \sin\overline{s-c}}$.

4° To compute the formula $\sin^2\frac{c}{2} = \sin\left(\frac{a+b}{2} + M\right) \cdot \sin\left(\frac{a+b}{2} - M\right)$; when $a = 84° \, 14' \, 29''$; $b = 44° \, 13' \, 45''$.

R. $c = 51° \, 6' \, 11''.33$.

5° To compute the formulæ $\sin a = \frac{N'}{\sin B . \sin C}$; $\sin b = \frac{N'}{\sin A . \sin C}$; $\sin c = \frac{N'}{\sin B . \sin A}$; when $A = 44° \, 10' \, 40''$; $B = 33° \, 22' \, 45''$; $C = 119° \, 55' \, 6''$.

R. $a = 50° \, 54' \, 30''.8$

$b = 37° \, 47' \, 18''$

$c = 74° \, 51' \, 49''$.

6° To compute the formulæ $\tan \frac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \cdot \tan \frac{c}{2}$; when $A = 130° \, 5' \, 22''$; $B = 32° \, 26' \, 6''$; and $c = 51° \, 6' \, 12''$.

R. $\begin{cases} a = 84° \, 14' \, 28'' \\ b = 44° \, 13' \, 13''. \end{cases}$

It seems useless to insert more such examples, as they can be found in the several systems of Astronomy in abundance, particularly in the excellent one by Professor Woodhouse, *vol.* 1, *second edition*, which no student, who desires to become intimately acquainted with Plain Astronomy, should fail to possess.

If the reader is content with a merely elementary knowledge of Trigonometry, such as would enable him to pass with facility through the popular parts of Plain Astronomy, to have read thus far, will be amply sufficient. If he wish, however, to be able to apply it to more minute and difficult researches, or if he wish merely for its own sake, and for the intellectual satisfaction that it affords to become acquainted with its extent and its power, he will be apt to consider the preceding part in no other light than as an

introduction, certainly an essential one, to more ample and extended research.

Before we proceed to the second part, we shall shew the use of Spherical Trigonometry in solving some Theorems and Problems respecting the sphære.

This subject, though (like the similar application of Plane Trigonometry) not essentially connected with any part of our treatise, must be allowed to be one of considerable interest, and which the more experienced student will find frequently of use.

CHAP. VII.

APPLICATION OF SPHERICAL TRIGONOMETRY TO THE SOLUTION OF THEOREMS AND PROBLEMS CONCERNING THE SPHÆRE AND SPHERICAL TRIANGLES.

1. *Required an expression for the cosine of an arc on the surface of a sphære, in terms of the cosines of the arcs between its extremities and three points on the surface, each the pole of a great circle through the other two.*

Let oo' be the arc in (Fig. 14), and x, y, z, the three points; then

$$\cos oo = \cos oy.\ \cos o'y + \sin oy.\ \sin o'y.\ \cos vv'$$

$$= \cos oy.\ \cos o'y + \cos ov.\ \cos o'v'.\ \sin(xv' + vz)$$

$$= \cos oy.\ \cos o'y + \cos ov.\ \cos o'v'.\left\{\cos vz.\ \cos v'z + \cos vx.\ \cos v'x\right\}$$

but by Napier's rules for circular parts $\cos ov.\ \cos vz = \cos oz$; $\cos o'v'.\ \cos v'z = \cos o'z$, &c. $\therefore$ we have

$$\cos oo' = \cos oy.\ \cos o'y + \cos ox.\ \cos o'x + \cos oz.\ \cos o'z \qquad (1)$$

If the arc $oo' = 90°$, we have

$$0 = \cos oy . \cos o'y + \cos ox . \cos o'x + \cos oz . \cos o'z \qquad (2)$$

If the arc should vanish, we have

$$1 = \cos^2 ox + \cos^2 oy + \cos^2 oz \qquad (3)$$

These three Theorems are of considerable and very frequent use in Mechanics.

The Theorem (1), *otherwise demonstrated*, as it might easily be, would serve as a very elegant basis of a system of Spherical Trigonometry, as we might by reversing the proof above given, arrive at an expression for the cosine of an angle in terms of the sides. This would require the consideration of axes, and thus interfere very considerably with the natural order of instruction.

2. *To find an expression for the area of a spherical triangle in terms of two sides, and the included angle.*

Denoting the area by S, we have already had $S = A + B + C - \pi$, $\therefore \tan \frac{1}{2}(A+B+C) = -\cot \frac{1}{2} S$; but by one of Napier's analogies, we have $\tan \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cdot \cot \frac{C}{2}$ $\therefore$ by formula (5), (Chap. II.), we have $\cot \frac{1}{2} S = \frac{\cot \frac{1}{2} a . \cot \frac{1}{2} b + \cos C}{\sin C}$

Hence with the same vertical angle, two spherical triangles are equal, that have the tangents of their half sides reciprocally proportional.

And the tangent of the half side of an equal isosceles triangle is a mean proportional between the tangents of the half sides of any others.

M. Legendre shews from the formula above given, how to determine the maximum triangle that can be formed, when we have given two sides; his method is as follows:

With rad. $OZ = 1$, (Fig. 15.) describe the semicircle VMZ, make the arc $ZX = C$; and on the other side of the centre take $OP = \cot \frac{1}{2} a . \cot \frac{1}{2} b$; join PX and let fall XY perpendicular to PZ.

In the triangle PXY we have $\cot P = \frac{PY}{XY} = \frac{\cot \frac{1}{2}a. \cot \frac{1}{2}b + \cos C}{\sin C}$; $\therefore P = \frac{1}{2}S$ $\therefore$ the surface S will be a max. when the angle P is so; and this will be the case when the angle P becomes MPO, which gives us $MPO = MOZ - \frac{1}{2}\pi = C - \frac{1}{2}\pi \therefore C = A + B$.

If the point P fall within the circle, there will be no *maximum*. This gives us $\cot \frac{1}{2}a. \cot \frac{1}{2}b. < 1$, whence $\cot \frac{1}{2}a. < \tan \frac{1}{2}b \therefore \tan(\frac{1}{2}\pi - \frac{1}{2}a) < \tan \frac{1}{2}b. \therefore \pi < a + b$.

3. *To find an expression for the area of a spherical triangle in terms of the three sides.*

We have already had

$$\cos \tfrac{1}{2}(A + B + C) = -\frac{\sqrt{\sin s. \sin(s-a). \sin(s-b) \sin(s-c)}}{2. \cos \frac{1}{2}a. \cos \frac{1}{2}b. \cos \frac{1}{2}c}$$

but

$$\cos \tfrac{1}{2}(A+B+C) = \cos(\tfrac{1}{2}S - \tfrac{1}{2}\pi) = -\sin \tfrac{1}{2}S \therefore$$

$$\sin \tfrac{1}{2}S = \frac{\sqrt{\sin s. \sin(s-a). \sin(s-b). \sin(s-c)}}{2. \cos \frac{1}{2}a. \cos \frac{1}{2}b. \cos \frac{1}{2}c} \qquad (1)$$

An expression for $\cot \frac{1}{2}S$ may be deduced in terms of the sides from the formula $\cot \frac{1}{2}S = \frac{\cot \frac{1}{2}a. \cot \frac{1}{2}b + \cos C}{\sin C}$. We have

$$\frac{1+\cos a}{\sin a} = \cot \tfrac{1}{2}a; \; \frac{1+\cos b}{\sin b} = \cot \tfrac{1}{2}b; \; \cos C = \frac{\cos c - \cos a. \cos b}{\sin a. \sin b};$$

and

$$\sin C = \frac{2}{\sin b. \sin c} \sqrt{\sin s. \sin(s-a). \sin(s-b). \sin(s-c)}.$$

Uniting all these values, we have

$$\cot \tfrac{1}{2}S = \frac{1 + \cos a + \cos b + \cos c}{2. \sqrt{\sin s. \sin(s-a). \sin(s-b). \sin(s-c)}} \qquad (2)$$

Multiplying together formulæ (1) and (2) we obtain a third formula.

$$\cos\tfrac{1}{2}S = \frac{1+\cos a+\cos b+\cos c}{4.\cos\frac{1}{2}a.\cos\frac{1}{2}b.\cos\frac{1}{2}c} = \frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a.\cos\frac{1}{2}b.\cos\frac{1}{2}c} \qquad (3)$$

A fourth formula, more elegant still than any of the preceding, may be deduced as follows:

By the formula $\tan\frac{1}{2}A = \dfrac{1-\cos A}{\sin A}$ we have $\tan\frac{1}{2}S = \dfrac{1-\cos\frac{1}{2}S}{\sin\frac{1}{2}S}$, whence we have

$$\tan\tfrac{1}{2}S = \frac{1-\cos^2\frac{1}{2}a-\cos^2\frac{1}{2}\beta-\cos^2\frac{1}{2}c+2.\cos\frac{1}{2}a.\cos\frac{1}{2}b.\cos\frac{1}{2}c}{\sqrt{\sin s.\sin(s-a).\sin(s-b).\sin(s-c)}}$$

The numerator of this fraction is the product of $\sin\frac{1}{2}a.\sin\frac{1}{2}c+\cos\frac{1}{2}b-\cos\frac{1}{2}a.\cos\frac{1}{2}c$, and $\sin\frac{1}{2}a.\sin\frac{1}{2}c-\cos\frac{1}{2}b+\cos\frac{1}{2}a.\cos\frac{1}{2}c$, or of $\cos\frac{1}{2}b-\cos\frac{1}{2}(a+c)$ and $\cos\frac{1}{2}(a-c)-\cos\frac{1}{2}b$, $\therefore$ by decomposing these latter into factors we obtain

$$\tan\tfrac{1}{2}S = \frac{4.\sin\frac{s}{2}.\sin\frac{s-a}{2}.\sin\frac{s-b}{2}.\sin\frac{s-c}{2}}{\sqrt{\sin s.\sin(s-a).\sin(s-b).\sin(s-c)}}$$

from which, by the formula $\sqrt{\frac{1}{2}\tan\frac{1}{2}A} = \dfrac{\sin\frac{1}{2}A}{\sqrt{\sin A}}$, we have

$$\tan\tfrac{1}{4}S = \sqrt{\tan\frac{s}{2}.\tan\frac{s-a}{2}.\tan\frac{s-b}{2}.\tan\frac{s-c}{2}} \qquad (4)$$

The proof here given of this very elegant formula does not differ in substance from that given by M. Legendre in his Geometry.—(*Note* x. *p.* 317, *Edit*, 11.) The formula itself is due to Simon Lhuilier.

4. ***Being given the three sides of a spherical triangle, to determine the position of the pole of the circumscribing circle.***

In (Fig. 16.) denoting any one of the three equal arcs AI, BI, CI, by the character φ, and the angle ACI by x, we have

$$\cos x = \frac{\cos\varphi - \cos b . \cos\varphi}{\sin b . \sin\varphi} = \frac{1-\cos b}{\sin b} . \cot\varphi = \frac{\sin b}{1+\cos b} . \cot\varphi$$

$$\cos(C-x) = \frac{1-\cos a}{\sin a} . \cot\varphi$$

$$\therefore \frac{\cos(C-x)}{\cos x} = \cos C + \sin C . \tan x = \frac{(1+\cos b) . (1-\cos a)}{\sin a . \sin b}$$

Replacing $\cos C$ and $\sin C$ by their values in terms of the sides, and denoting the double radical in the value of $\sin C$ by M, we obtain

$$\tan x = \frac{1+\cos b - \cos a - \cos c}{M}$$

From the equality of the arcs AI, BI, CI, x is obviously equal to $\frac{1}{2}(A+C-B)$; and as similar formulæ and similar expressions are true for the other angles ICB, IBA, we have the following formulæ:

$$\tan\tfrac{1}{2}(A+C-B) = \frac{1+\cos b - \cos a - \cos c}{M}$$

$$\tan\tfrac{1}{2}(B+C-A) = \frac{1+\cos a - \cos b - \cos c}{M}$$

$$\tan\tfrac{1}{2}(A+B-C) = \frac{1+\cos c - \cos a - \cos b}{M}$$

To these we may add, from the preceding proposition,

$$\tan\tfrac{1}{2}(A+B+C) = \frac{-1-\cos a - \cos b - \cos c}{M}$$

Let us now proceed to find an expression for $\tan\varphi$. We have, (from the value found for $\tan x$),

$$1+\tan^2 x, \text{ or } \frac{1}{\cos^2 x}=\frac{2(1+\cos b)(1-\cos c)(1-\cos a)}{M^2}$$

$$=\frac{16.\cos^4\frac{1}{2}b.\sin^2\frac{1}{2}c.\sin^2\frac{1}{2}a}{M^2}$$

but from the equation $\cos x=\frac{1-\cos b}{\sin b}.\cot\varphi$, we have

$$\tan\tfrac{1}{2}\varphi=\frac{\tan\frac{1}{2}b}{\cos x}\ \therefore\ \tan\varphi=\frac{4.\sin\frac{1}{2}a.\sin\frac{1}{2}b.\sin\frac{1}{2}c}{M}$$

$$=\frac{2.\sin\frac{1}{2}a.\sin\frac{1}{2}b.\sin\frac{1}{2}c}{\sqrt{\sin s.\sin(s-a).\sin(s-b).\sin(s-c)}} \qquad (1)$$

Hence we have

$$\sin\varphi=\frac{4.\sin\frac{1}{2}a.\sin\frac{1}{2}b.\sin\frac{1}{2}c}{\sqrt{M^2+16.\sin^2\frac{1}{2}a.\sin^2\frac{1}{2}b.\sin^2\frac{1}{2}c}} \qquad (2)$$

$$\cos\varphi=\frac{M}{\sqrt{M^2+16.\sin^2\frac{1}{2}a.\sin^2\frac{1}{2}b.\sin^2\frac{1}{2}c}} \qquad (3)$$

These expressions differently demonstrated, the reader will find in Lagrange's Memoir, *Journal de l'Ecole Polytechnique*, or in Leybourne's Repository, vol 5, part 1. They are here demonstrated after the manner of M. Legendre.

5. *Being given the three sides of a spherical triangle, to determine the position of the pole of the inscribed circle.*

In (Fig. 17.) from the equality of the arcs OR, OP, OQ, we have $BR=s-b$, $CQ=s-c$, $AQ=s-a$. But by the fourth formula in Case 1, we have

$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-a).\sin(s-c)}{\sin s.\sin(s-b)}}$$

and by Napier's rules, tan OP = tan QBP. sin BP, whence denoting OP by θ, we obtain

$$\tan\theta = \sqrt{\frac{\sin(s-a).\sin(s-b).\sin(s-c)}{\sin s}}.$$

The reader will find no difficulty in perceiving the truth of the following expressions:

$$\tan\theta' = \sqrt{\frac{\sin s.\sin(s-a).\sin(s-c)}{\sin(s-b)}}$$

$$\tan\theta'' = \sqrt{\frac{\sin s.\sin(s-b).\sin(s-c)}{\sin(s-a)}}$$

$$\tan\theta''' = \sqrt{\frac{\sin s.\sin(s-a).\sin(s-b)}{\sin(s-c)}}$$

Where θ', denotes the *circular* radius of the circle that touches b, and a, c, produced; θ'' that of the circle touching a, and b, c, produced; and θ''' that of the circle touching c, and a, b, produced.

6. *Given base and area of a spherical triangle, to find the locus of its vertex.*

Let ABC (Fig. 18.) be a spherical triangle, whose base is AB, and vertex C. Erect a perpendicular IPK from the middle point of AB, take P, the pole of AB, and draw PCD. Denoting ID by p, and DC by q, we have from the right angled triangles ACD, BCD; $\cos a = \cos q.\cos(p-\frac{1}{2}c)$; $\cos b = \cos q.\cos(p+\frac{1}{2}c)$. We have already had

$$\cot\tfrac{1}{2}S = \frac{1+\cos a+\cos b+\cos c}{\sin a.\sin b.\sin C}$$

Making in this the following substitutions, viz. for $\cos a + \cos b$, writing $2\cos q.\cos p.\cos\frac{1}{2}c$; for $1+\cos c$, writing $2\cos^2\frac{1}{2}c$; and for $\sin b.\sin C$, writing $\sin c.\sin B$. or $2\sin\frac{1}{2}c.\cos\frac{1}{2}c.\sin B$; then

$$\cot\tfrac{1}{2}S = \frac{\cos\frac{1}{2}c+\cos p.\cos q}{\sin\frac{1}{2}c.\sin q}$$

but in the right angled triangle BCD we have $\sin a.\ \sin B = \sin q$,

$\therefore \cot \tfrac{1}{2}S = \dfrac{\cos \tfrac{1}{2}c + \cos p.\ \cos q}{\sin \tfrac{1}{2}c.\ \sin q}$, or $\cos p.\ \cos f = \cot \tfrac{1}{2}S.\ \sin \tfrac{1}{2}c.$ $\sin q - \cos \tfrac{1}{2}c$, which equation shews the relation between p and q that determines the locus.

Taking on the prolongation of IP, a line $PK = x$, such that $\cot x = \cot \tfrac{1}{2}$ S. $\sin \tfrac{1}{2}c$; joining KC and calling it y; we then have $\cos y = \sin q.\ \cos x - \sin x.\ \cos q.\ \cos p.$ in which substituting for $\cos p.\ \cos q$ its value, $\cot \tfrac{1}{2}$ S. $\sin \tfrac{1}{2}c.\ \sin q - \cos \tfrac{1}{2}c$, we obtain

$$\cos y = \sin x.\ \cos \tfrac{1}{2}c + \sin q.\ (\cos x - \sin a.\ \cot \tfrac{1}{2}s.\ \sin \tfrac{1}{2}c)$$

which, from the value that has been assumed for x, gives us $\cos y =$ $\sin x.\ \cos \tfrac{1}{2}c$, a constant quantity. Hence the locus required is the arc of a small circle, the pole of which is K, and the circular radius y.

This Theorem is due to Lexell.—*(Vide tom* v. *part* 1. *Nova acta Acad. Petrop.)*

Having now gone through the principles of the general theory of sines, together with two of its most important applications, it may not be useless to recapitulate our results in a few tables, after the manner of Cagnoli. The number of the formulæ might be much increased, but, perhaps, without any adequate advantage.

TRIGONOMETRICAL TABLES.

The following Tables the reader will find convenient for the purpose of reference, as well as for exercise. The proofs of them may be had from the second chapter of this work.

TABLE I.

The most important Analytical Values of Trigonometrical Lines.	
Values of sin A.	8. $\dfrac{2\tan\frac{1}{2}A}{1+\tan^2\frac{1}{2}A}$
1. FORMULA...$\cos A\tan A$	9. $\dfrac{2}{\cot\frac{1}{2}A+\tan\frac{1}{2}A}$
2. $\dfrac{\cos A}{\cot A}$	10. $\dfrac{\sin(30°+A)-\sin(30°-A)}{\sqrt{3}}$
3. $\sqrt{(1-\cos^2 A)}$	11. $2\sin^2(45°+\frac{1}{2}A)-1$
4. $\dfrac{1}{\sqrt{(1+\cot^2 A)}}$	12. $1-2\sin^2(45°+\frac{1}{2}A)$
5. $\dfrac{\tan A}{\sqrt{(1+\tan^2 A)}}$	13. $\dfrac{1-\tan^2(45°-\frac{1}{2}A)}{1+\tan^2(45°-\frac{1}{2}A)}$
6. $2\sin\frac{1}{2}A\cos\frac{1}{2}A$	14. $\dfrac{\tan(45°+\frac{1}{2}A)-\tan(45°-\frac{1}{2}A)}{\tan(45°+\frac{1}{2}A)+\tan(45°-\frac{1}{2}A)}$
7. $\sqrt{\dfrac{1-\cos 2A}{2}}$	15. $\sin(60°+A)-\sin(60°-A)$

Values of cos *A*.

16. $\dfrac{\sin A}{\tan A}$

17. $\sin A \cot A$

18. $\sqrt{(1-\sin^2 A)}$

19. $\dfrac{1}{\sqrt{(1+\tan^2 A)}}$

20. $\dfrac{\cot A}{\sqrt{(1+\cot^2 A)}}$

21. $\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A$

22. $1 - 2\sin^2 \frac{1}{2}A$

23. $2\cos^2 \frac{1}{2}A - 1$

24. $\sqrt{\dfrac{1+\cos 2A}{2}}$

25. $\dfrac{1-\tan^2 \frac{1}{2}A}{1+\tan^2 \frac{1}{2}A}$

26. $\dfrac{\cot \frac{1}{2}A - \tan \frac{1}{2}A}{\cot \frac{1}{2}A + \tan \frac{1}{2}A}$

27. $\dfrac{1}{1+\tan A \tan \frac{1}{2}A}$

28. $\dfrac{2}{\tan(45^\circ + \frac{1}{2}A) + \cot(45^\circ + \frac{1}{2}A)}$

29. $2\cos(45^\circ + \frac{1}{2}A).\cos(45^\circ - \frac{1}{2}A)$

30. $\cos(60^\circ + A) + \cos(60^\circ - A)$

Values of tan *A*.

31. $\dfrac{\sin A}{\cos A}$

32. $\dfrac{1}{\cot A}$

33. $\sqrt{\left(\dfrac{1}{\cos^2 A} - 1\right)}$

34. $\dfrac{\sin A}{\sqrt{(1-\sin^2 A)}}$

35. $\dfrac{\sqrt{(1-\cos^2 A)}}{\cos A}$

36. $\dfrac{2\tan \frac{1}{2}A}{1-\tan^2 \frac{1}{2}A}$

37. $\dfrac{2\cot \frac{1}{2}A}{\cot^2 \frac{1}{2}A - 1}$

38. $\dfrac{2}{\cot \frac{1}{2}A - \tan \frac{1}{2}A}$

39. $\cot A - 2\cot 2A$

40. $\dfrac{1-\cos 2A}{\sin 2A}$

41. $\dfrac{\sin 2A}{1+\cos 2A}$

42. $\sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$

43. $\dfrac{\tan(45^\circ + \frac{1}{2}A) - \tan(45^\circ - \frac{1}{2}A)}{2}$

TABLE II.

Trigonometrical formulæ relative to two arcs.

1. $\sin(A+B)=\sin A.\cos B+\cos A.\sin B$
2. $\sin(A-B)=\sin A.\cos B-\cos A.\sin B$
3. $\cos(A+B)=\cos A.\cos B-\sin A.\sin B$
4. $\cos(A-B)=\cos A.\cos B+\sin A.\sin B$
5. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A.\tan B}$
6. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A.\tan B}$
7. $\left.\begin{matrix}\sin(45^\circ\pm B)=\\ \cos(45^\circ\pm B)=\end{matrix}\right\}\dfrac{\cos B\pm\sin B}{\sqrt{2}}$
8. $\tan(45^\circ\pm B)=\dfrac{1\pm\tan B}{1\mp\tan B}$
9. $\tan^2(45^\circ\pm\frac{1}{2}B)=\dfrac{1\pm\sin B}{1\mp\sin B}$
10. $\tan(45^\circ\pm\frac{1}{2}B)=\dfrac{1\pm\sin B}{\cos B}=\dfrac{\cos B}{1\mp\sin B}$
11. $\dfrac{\sin(A+B)}{\sin(A-B)}=\dfrac{\tan A+\tan B}{\tan A-\tan B}$
12. $\dfrac{\cos(A+B)}{\cos(A-B)}=\dfrac{\cot B-\tan A}{\cot B+\tan A}=\dfrac{\cot A-\tan B}{\cot A+\tan B}$
13. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A+B)}$
14. $\dfrac{\cos B+\cos A}{\cos B-\cos A}=\dfrac{\cot\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$
15. $\sin A.\cos B=\frac{1}{2}.\sin(A+B)+\frac{1}{2}.\sin(A-B)$
16. $\cos A.\sin B=\frac{1}{2}.\sin(A+B)-\frac{1}{2}.\sin(A-B)$
17. $\sin A.\sin B=\frac{1}{2}.\cos(A-B)-\frac{1}{2}.\cos(A+B)$
18. $\cos A.\cos B=\frac{1}{2}.\cos(A+B)+\frac{1}{2}.\cos(A-B)$
19. $\sin A+\sin B=2.\sin\frac{1}{2}(A+B).\cos\frac{1}{2}(A-B)$
20. $\cos A+\cos B=2.\cos\frac{1}{2}(A+B).\cos\frac{1}{2}(A-B)$
21. $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A.\cos B}$
22. $\cot A+\cot B=\dfrac{\sin(A+B)}{\sin A.\sin B}$
23. $\sin A-\sin B=2.\cos\frac{1}{2}(A+B).\sin\frac{1}{2}(A-B)$
24. $\cos B-\cos A=2.\sin\frac{1}{2}(+B).\sin\frac{1}{2}(A-B)$
25. $\tan A-\tan B=\dfrac{\sin(A-B)}{\cos A.\cos B}$
26. $\cot B-\cot A=\dfrac{\sin(A-B)}{\sin A.\sin B}$
27. $\left.\begin{matrix}\sin^2 A-\sin^2 B=\\ \cos^2 B-\cos^2 A=\end{matrix}\right\}\sin(A+B).\sin(A-B)$
28. $\cos^2 A-\sin^2 B=\cos(A-B).\cos(A+B)$
29. $\tan^2 A-\tan^2 B=\dfrac{\sin(A+B).\sin(A-B)}{\cos^2 A.\cos^2 B}$
30. $\cot^2 B-\cot^2 A=\dfrac{\sin(A-B).\sin(A+B)}{\sin^2 A.\sin^2 B}$

TABLE III.

Formulæ for the resolution of a right lined triangle.	
Values of c.	*Values of b.*
1. $\frac{a.\ \sin C}{\sin A}$	10. $\frac{c.\ \sin B}{\sin C}$
2. $\frac{b.\ \sin C}{\sin B}$	11. $\frac{a.\ \sin B}{\sin A}$
3. $\frac{a}{\cos B+\sin B.\ \cot C}$	12. $\frac{c}{\cos A+\sin A.\ \cot B}$
4. $\frac{b}{\cos A+\sin A.\ \cot C}$	13. $\frac{a}{\cos C+\sin C.\ \cot B}$
5. $a.\ \cos B+a.\ \sin B.\ \cot A$	14. $c.\ \cos A+c.\ \sin A\ \cot C$
6. $b.\ \cos A+b.\ \sin A.\ \cot B$	15. $a.\ \cos C+a.\ \sin C.\ \cot A$
7. $\sqrt{a^2+b^2-2ab.\ \cos C}$	16. $\sqrt{a^2+c^2-2ac.\ \cos B}$
8. $a.\ \cos B\pm\sqrt{b^2-a^2.\ \sin^2 B}$	17. $c.\ \cos A\pm\sqrt{a^2-c^2.\ \sin^2 A}$
9. $b.\ \cos A\pm\sqrt{a^2-b^2.\ \sin^2 A}$	18. $a.\ \cos C\pm\sqrt{c^2-a^2.\ \sin^2 C}$

Continuation of Table III.

Values of a	*Values of* sin *A*.
19. $\frac{b.\sin A}{\sin B}$	28. $\frac{a.\sin C}{c}$
20. $\frac{c.\sin A}{\sin C}$	29. $\frac{a.\sin B}{b}$
21. $\frac{b}{\cos C+\sin C.\cot A}$	30. $\sin(B+C)$
22. $\frac{c}{\cos B+\sin B.\cot A}$	31. $\sin B.\cos C+\cos B.\sin C$
23. $b.\cos C+b.\sin C.\cot B$	32. $\frac{a.\sin C}{\sqrt{a^2+b^2-2ab.\cos C}}$
24. $c.\cos B+c.\sin B.\cot C$	33. $\frac{a.\sin B}{\sqrt{a^2+c^2-2ac.\cos B}}$
25. $\sqrt{c^2+b^2-2bc.\cos A}$	34. $\sqrt{1-\left(\frac{c^2+b^2-a^2}{2bc}\right)^2}$
26. $b.\cos C\pm\sqrt{c^2-b^2\sin^2 C}$	35. $\frac{\sin C.\left\{b.\cos C\pm\sqrt{c^2-b^2.\sin^2 C}\right\}}{c}$
27. $c.\cos B\pm\sqrt{b^2-c^2.\sin^2 B}$	36. $\frac{\sin B.\left\{c.\cos B\pm\sqrt{b^2-c^2.\sin^2 B}\right\}}{b}$

Values of cos *A.*	*Values of* tan *A.*
37. $\dfrac{\pm\sqrt{c^2-a^2.\sin^2 C}}{c}$	46. $\dfrac{a.\sin C}{\pm\sqrt{c^2-a^2.\sin^2 C}}$
38. $\dfrac{\pm\sqrt{b^2-a^2.\sin^2 B}}{b}$	47. $\dfrac{a.\sin B}{\pm\sqrt{b^2-a^2.\sin^2 B}}$
39. $-\cos(B+C)$	48. $-\tan(B+C)$
40. $\sin B.\sin C-\cos B.\cos C$	49. $\dfrac{\tan B+\tan C}{\tan B.\tan C-1}$
41. $\dfrac{b-a.\cos C}{\sqrt{a^2+b^2-2ab.\cos C}}$	50. $\dfrac{a.\sin C}{b-a.\cos C}$
42. $\dfrac{c-a.\cos B}{\sqrt{a^2+c^2-2ac.\cos B}}$	51. $\dfrac{a.\sin B}{c-a.\cos B}$
43. $\dfrac{c^2+b^2-a^2}{2bc}$	52. $\pm\sqrt{\dfrac{4b^2c^2}{(b^2+c^2-a^2)^2}-1}$
44. $\dfrac{b.\sin^2 C\mp\cos C\sqrt{c^2-b^2.\sin^2 C}}{c}$	53. $\dfrac{b.\cos C\pm\sqrt{c^2-b^2.\sin^2 C}}{b.\sin C\mp\cot C\sqrt{c\ -b^2.\sin^2 C}}$
45. $\dfrac{c.\sin^2 B\mp\cos B\sqrt{b^2-c^2.\sin^2 B}}{b}$	54. $\dfrac{c.\cos B\pm\sqrt{b^2-c^2.\sin^2 B}}{c.\sin B\mp\cot B\sqrt{b^2-c^2.\sin^2 B}}$

TABLE IV.

Formulæ for the resolution of spherical triangles.

1. $\sin A = \dfrac{\sin a.\ \sin C}{\sin c} = \dfrac{\sin a.\ \sin B}{\sin b}$

2. $\sin B = \dfrac{\sin b.\ \sin A}{\sin a} = \dfrac{\sin b.\ \sin C}{\sin c}$

3. $\sin C = \dfrac{\sin c.\ \sin B}{\sin b} = \dfrac{\sin c.\ \sin A}{\sin a}$

4. $\cos A = \dfrac{\cos a - \cos b.\ \cos c}{\sin b.\ \sin c} = \cos a.\ \sin B.\ \sin C - \cos B.\ \cos C$

5. $\cos B = \dfrac{\cos b - \cos a.\ \cos c}{\sin a.\ \sin c} = \cos b.\ \sin A.\ \sin C + \cos A.\ \cos C$

6. $\cos C = \dfrac{\cos c - \cos a.\ \cos b}{\sin a.\ \sin b} = \cos c.\ \sin A.\ \sin B + \cos A.\ \cos B$

7. $\tan A = \dfrac{\sin B}{\sin c.\ \cot a - \cos c.\ \cos B} = \dfrac{\sin C}{\sin b.\ \cot a - \cos b.\ \cos C}$

8. $\tan B = \dfrac{\sin C}{\sin a.\ \cot b - \cos a.\ \cos C} = \dfrac{\sin A}{\sin c.\ \cot b - \cos c.\ \cos A}$

9. $\tan C = \dfrac{\sin A}{\sin b.\ \cot c - \cos b.\ \cos A} = \dfrac{\sin B}{\sin a.\ \cot c - \cos a.\ \cos B}$

10. $\sin a = \dfrac{\sin c.\ \sin A}{\sin C} = \dfrac{\sin b.\ \sin A}{\sin B}$

11. $\sin b = \dfrac{\sin a.\ \sin B}{\sin A} = \dfrac{\sin c.\ \sin B}{\sin C}$

12. $\sin c = \dfrac{\sin b.\ \sin C}{\sin B} = \dfrac{\sin a.\ \sin C}{\sin A}$

Continuation of Table IV.

13. $\cos a = \frac{\cos A + \cos B.\cos C}{\sin B.\sin C} = \cos A.\sin c.\sin b + \cos c.\cos b.$

14. $\cos b = \frac{\cos B + \cos A.\cos C}{\sin A.\sin C} = \cos B.\sin a.\sin c + \cos a.\cos c.$

15. $\cos c = \frac{\cos C + \cos A.\cos B}{\sin A.\sin B} = \cos C.\sin b.\sin a + \cos b.\cos a$

16. $\tan a = \frac{\sin c}{\sin B.\cot A + \cos B.\cos c} = \frac{\sin b}{\sin C.\cot A + \cos C.\cos a}$

17. $\tan b = \frac{\sin a}{\sin C.\cot B + \cos C.\cos a} = \frac{\sin c}{\sin A.\cot B + \cos A.\cos c}$

18. $\tan c = \frac{\sin b}{\sin A.\cot C + \cos A.\cos b} = \frac{\sin a}{\sin B.\cot C + \cos B.\cos a}$

19. $\sin A = \frac{2}{\sin b.\sin c}\sqrt{\sin s.\sin(s-a).\sin(s-b).\sin(s-c)}$

20. $\cos\frac{A}{2} = \sqrt{\frac{\sin s.\sin(s-a)}{\sin b.\sin c}}$

21. $\sin\frac{A}{2} = \sqrt{\frac{\sin(s-a).\sin(s-b)}{\sin b.\sin c}}$

22. $\tan\frac{A}{2} = \sqrt{\frac{\sin(s-c).\sin(s-b)}{\sin s.\sin(s-a)}}$

23. $\sin a = \frac{2}{\sin B.\sin C}\sqrt{-\cos S'.\cos(S'-A).\cos(S'-B).\cos(S'-C)}$

24. $\sin\frac{a}{2} = \sqrt{\frac{-\cos S'.\cos(S'-A}{\sin B.\sin C}}$

Continuation of Table IV.

25. $\cos\frac{a}{2} = \sqrt{\frac{\cos(S'-B).\cos(S'-C)}{\sin B.\sin C}}$

26. $\tan\frac{a}{2} = \sqrt{\frac{-\cos S'.\cos(S'-A)}{\cos(S'-B).\cos(S'-C)}}$

27. $\tan\frac{1}{2}(A+B) = \cot\frac{1}{2}C.\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}$

28. $\tan\frac{1}{2}(A-B) = \cot\frac{1}{2}C.\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}$

29. $\tan\frac{1}{2}(a+b) = \tan\frac{1}{2}c.\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$

30. $\tan\frac{1}{2}(a-b) = \tan\frac{1}{2}c.\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$

PART II.

CHAP. I.

ON THE SUMMATION OF SERIES BY TRIGONOMETRICAL ARTIFICE.

1. *To sum the series* $\cos A + \cos(A+B) + \cos(A+2B) \ldots\ldots\ldots \cos(A+(n-1)B)$

The several following formulæ are obviously true by formula (10) Chap. II. Part I.

$$\sin(A+\tfrac{1}{2}B) - \sin(A-\tfrac{1}{2}B) = 2.\cos A.\sin\tfrac{1}{2}B$$

$$\sin(A+\tfrac{3}{2}B) - \sin A+\tfrac{1}{2}B) = 2.\cos(A+B).\sin\tfrac{1}{2}B$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad = \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\sin\left(A+\frac{2n-1}{2}.B\right) - \sin\left(A+\frac{2n-3}{2}\right) = 2.\cos(A+(n-1)B.)\sin\tfrac{1}{2}B$$

Adding together all these equations, the terms that compose the left hand members destroy each other, with the exception of the negative in the first equation, and the positive in the last; but the right hand members constitute the given series multiplied by $2.\sin\tfrac{1}{2}B$. Let us denote the series by the character Σ, then

$$\sin\left(A+\frac{2n-1}{2}.B\right) - \sin(A-\tfrac{1}{2}B) = 2.\sin\tfrac{1}{2}B.\Sigma$$

$$\therefore \Sigma = \frac{\cos\left(A+\frac{n-1}{2}.B\right).\sin\frac{n}{2}.B}{\sin\tfrac{1}{2}B}$$

If continued *ad inf.* the terms of the left hand members would continue to destroy, so that we would have

$$\Sigma = \frac{-\sin(A-\frac{1}{2}B)}{2.\sin\frac{1}{2}B}$$

If $B=A$, $\Sigma = \dfrac{\cos\frac{n+1}{2}A}{\sin\frac{1}{2}A}.\sin\frac{n}{2}A.$ If $nA=2\pi$, $\Sigma=0$

If $B=A$, Σ *ad inf.* $=-\frac{1}{2}$. If $\frac{1}{2}B=A$, Σ *ad inf.* $=0$.

2. *To sum the series* $\sin A - \sin(A+B) + \sin(A+2B)\ldots\ldots\ldots \pm \sin(A+(n-1)B)$.

By formula (9) Chap. II. Part I., we have the following equations:

$$\sin(A+\tfrac{1}{2}B)+\sin(A-\tfrac{1}{2}B) = 2.\sin A.\cos\tfrac{1}{2}B$$

$$-\sin(A+\tfrac{3}{2}B)-\sin(A+\tfrac{1}{2}B) = -2.\sin(A+B)\cos\tfrac{1}{2}B$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad = \quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\pm\sin\left(A+\frac{2n-1}{2}.B\right)\pm\sin\left(A+\frac{2n-3}{2}.B\right)=+2.\sin(A+nB).\cos\tfrac{1}{2}B$$

$\therefore \pm\sin\left(A+\dfrac{2n-1}{2}.B\right)+\sin(A-\frac{1}{2}B) = 2\cos\frac{1}{2}B.\Sigma.$ Using the character Σ, as before, to denote the sum of the series.

If we sum an even number of terms of the series, we obtain

$$\Sigma = \frac{\sin\left(A+\frac{n-1}{2}.B\right).\cos\frac{n}{2}B}{\cos\frac{1}{2}B}.$$

If an odd,

$$\Sigma = \frac{\cos\left(A+\frac{n-1}{2}.B\right).\sin\frac{n}{2}B}{\cos\frac{1}{2}B}.$$

If $B=A$,

$$\Sigma=\frac{\sin\frac{n+1}{2}.A.\cos\frac{n}{2}A}{\cos\frac{1}{2}A}, \text{ or } \Sigma=\frac{\cos\frac{n+1}{2}A.\sin\frac{n}{2}A}{\cos\frac{1}{2}A}.$$

If continued *ad inf.*,

$$\Sigma=\frac{\sin(A-\frac{1}{2}B)}{2.\cos\frac{1}{2}B}.$$

If $\frac{1}{2}B=A$, $\Sigma=0$, unless when $A=90°$, and then Σ is infinite, for the series becomes $1+1+1$, &c. *ad inf.*

If $B=A$, Σ *ad inf.* $=\frac{1}{2}.\tan\frac{1}{2}A$.

By the former of the two last remarks, we have the following series:

$$\left.\begin{array}{lr}\sin A-\sin 3A+\sin 5A-\&c.\ ad\ inf. & =0\\ -\sin 2A+\sin 6A-\sin 10A+\&c. & =0\\ \sin 3A-\sin 9A+\sin 15A-\&c. & =0\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots & =0\end{array}\right\}(A)$$

And by the latter of these remarks we have

$$\left.\begin{array}{lrl}\sin A-\sin 2A+\sin 3A-\&c.\ ad\ inf. & = & \frac{1}{2}.\tan\frac{1}{2}A\\ -\sin 3A+\sin 6A-\sin 9A+\&c. & =- & \frac{1}{2}.\tan\frac{3}{2}A\\ \sin 5A-\sin 10A+\sin 15A-\&c. & = & \frac{1}{2}.\tan\frac{5}{2}A\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots & = & \ldots\quad\ldots\end{array}\right\}(B)$$

The left hand members of the classes of equations (A) and (B) are identical, $\therefore$ $\tan\frac{1}{2}A-\tan\frac{3}{2}A+\tan\frac{5}{2}A-$&c. *ad inf.* $=0$.

3. *To sum the series* $\sin A+\sin(A+B)+\ldots\ldots\ldots\sin(A+(n-1)B)$.

By formula (12) Chap. II. Part I., we have the following equations.

$$\cos(A-\tfrac{1}{2}B)-\cos(A+\tfrac{1}{2}B)=2.\sin A.\sin\tfrac{1}{2}B$$

$$\cos(A+\tfrac{1}{2}B)-\cos(A+\tfrac{3}{2}B)=2.\sin(A+B).\sin\tfrac{1}{2}B$$

$$\cos(A+\tfrac{3}{2}B)-\cos(A+\tfrac{5}{2}B)=2.\sin(A+2B).\sin\tfrac{1}{2}B$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad=\quad\ldots\quad\ldots\quad\ldots$$

$$\cos\left(A+\frac{2n-3}{2}.B\right)-\cos\left(A+\frac{2n-1}{2}.B\right)=2.\sin(A+2B).\sin\tfrac{1}{2}B$$

$\therefore -\cos\left(A+\dfrac{2n-1}{2}.B\right)+\cos(A-\tfrac{1}{2}B)=2.\sin\tfrac{1}{2}B.\Sigma$, denoting the series as usual by the character Σ. Hence we have

$$\Sigma=\frac{\sin\left(A+\frac{n-1}{2}.B\right).\sin\frac{n}{2}B}{\sin\frac{1}{2}B}.$$

If $B=A$, $\Sigma=\dfrac{\sin\frac{n+1}{2}A}{\sin\frac{1}{2}A}.\sin\dfrac{n}{2}A.$ If $nA=2\pi$, $\Sigma=0$

If continued *ad inf.* $\Sigma=\dfrac{\cos(A-\frac{1}{2}B)}{2.\sin\frac{1}{2}B}$

If $\frac{1}{2}B=A$, Σ *ad inf.* $=\dfrac{1}{2.\sin A}$. If $B=A$, Σ *ad inf.* $=\frac{1}{2}.\cot\dfrac{A}{2}$

By the former of these last remarks we have

$$\sin A+\sin 3A+\sin 5A+\&c.\ \textit{ad inf.}=\frac{1}{2.\sin A}$$

$$\sin 2A+\sin 6A+\sin 10A+\&c.=\frac{1}{2.\sin 2A}$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad=\quad\ldots\quad\ldots$$

Summing which, vertically, we have by the latter of the two preceding remarks, combined with these equations,

$$\cot \tfrac{1}{2} A + \cot \tfrac{3}{2} A + \cot \tfrac{5}{2} A - \ldots. ad\ inf. = \text{cosec.} A + \text{cosec.}\ 2A \ldots. ad\ inf.$$

From the above we can demonstrate a very elegant Geometrical Theorem, which we shall have occasion for in the sequel.

If in a circle, a regular polygon of an odd number of sides be inscribed, and if lines be drawn from any point in the circumference to the several angles of the polygon, then denoting these lines in their order by the characters c', c'', c''', c^{iv}, &c., *it may be proved that* $c' + c''' + c^{v} + \&c. = c'' + c^{iv} + c^{vi}$, &c.

Let A be half the arc subtended by the side c', and B half that subtended by a side of the polygon, let the number of the sides of the polygon be $2n+1$, then $B = \frac{\pi}{2n+1}$, $\therefore c''' = 2. \sin (A + 2B) = 2. \sin \left(A + \frac{2\pi}{2n+1}\right)$. Hence half the sum of the odd chords which we shall call S, is represented by the series

$$\sin A + \sin \left(A + \frac{2\pi}{2n+1}\right) + \ldots\ldots\ldots\ldots \sin \left(A + \frac{2n\pi}{2n+1}\right).$$

$$\therefore S = \frac{\sin\left(A + \frac{n\pi}{2n+1}\right). \sin \frac{n+1}{2n+1}. \pi}{\sin \frac{\pi}{2n+1}}$$ by Art. 3 of this chapter.

Similarly we have S', (half the sum of the even chords)

$$= \frac{\sin \left(A + \frac{n\pi}{2n+1}\right). \sin \frac{n\pi}{2n+1}}{\sin \frac{\pi}{2n+1}},$$

and since $\sin \frac{n+1}{2n+1}. \pi = \sin \frac{n\pi}{2n+1}$, we have $S = S'$.

If the polygon be of an even number $(2n)$ of sides, then the series of semichords to the 1st, 3d, 5th, &c. angles would be

$\sin A + \sin\left(A+\frac{2\pi}{2n}\right)$ .... $\sin\left(A+(n-1).\frac{2\pi}{2n}\right)$; to the 2d, 4th, 6th angles, the series would be

$$\sin\left(A+\frac{\pi}{2n}\right)+\sin\left(\left(A+\frac{\pi}{2n}\right)+\frac{2\pi}{2n}\right).....\sin\left(\left(A+\frac{\pi}{2n}\right)+(n-1).\frac{2\pi}{2n}\right).$$

The sum of the former, $\Sigma = \dfrac{\sin\left(A+\frac{n-1}{2}.\frac{2\pi}{2n}\right).\sin\frac{n}{2}.\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$

The sum of the latter, $\Sigma' = \dfrac{\sin\left(A+\frac{\pi}{2}\right).\sin\frac{n}{2}.\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$

$\therefore$ the difference of the two systems of chords,

$$2(\Sigma-\Sigma') = 2.\frac{\cos\left(A-\frac{\pi}{2n}\right)-\cos A}{\sin\frac{\pi}{2n}} = 2.\sin\left(A-\frac{\pi}{4n}\right).\tan\frac{\pi}{4n}.$$

In (Fig. 16.) let o be the point whence the chords are drawn; $mvs=2B$; v the middle point of mvs; and $ovs=2A$; then the arc $ov = 2\left(A-\frac{\pi}{4n}\right)$ $\therefore$ the chord $ov = 2.\sin\left(A-\frac{\pi}{4n}\right)$ and $rv = \tan\frac{\pi}{4n}$, whence the difference of the two systems of chords is a 4th proportional to Cv, vr, and vo.

4. *To sum the series* $\cos A - \cos(A+B) + \cos(A+2B) - \&c. \pm \cos(A+(n-1)B)$.

By formula (11) Chap. II. Part I., we have the following equations:

$$\cos\left(A-\tfrac{1}{2}B\right)+\cos\left(A+\tfrac{1}{2}B\right)=2\cos A.\cos\tfrac{1}{2}B$$

$$-\cos\left(A+\tfrac{3}{2}B\right)-\cos\left(A+\tfrac{5}{2}B\right)=-2.\cos\left(A+B\right).\cos\tfrac{1}{2}B$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad=\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\pm\cos\left(A+\frac{2n-3}{2}\cdot B\right)\pm\cos\left(A+\frac{2n-1}{2}\cdot B\right)=\pm 2.\cos\left(A+(n-1)B\right).\cos\tfrac{1}{2}B$$

$$\therefore\ \cos\left(A-\tfrac{1}{2}B\right)\pm\cos\left(A+\frac{2n-1}{2}\cdot B\right)=2.\cos\tfrac{1}{2}B.\ \Sigma.$$

Hence $\Sigma=\dfrac{\cos\left(A+\frac{n-1}{2}\cdot B\right).\cos\frac{n}{2}B}{\cos\frac{1}{2}B}$, if an odd number of the terms of the series be summed; and $\Sigma=\dfrac{\sin\left(A+\frac{n-1}{2}\cdot B\right).\sin\frac{n}{2}B}{\sin\frac{1}{2}B}$, if an even number.

If $B=A$,

$$\Sigma=\frac{\cos\frac{n+1}{2}A.\ \cos\frac{n}{2}A}{\cos\frac{1}{2}A},\ \text{or}=\frac{\sin\frac{n+1}{2}A.\ \sin\frac{n}{2}A}{\cos\frac{1}{2}A}$$

If continued *ad inf.* $\Sigma=\dfrac{\cos\left(A-\frac{1}{2}B\right)}{2\cos\frac{1}{2}B}$

If $\frac{1}{2}B=A$, $\Sigma=\dfrac{1}{2\cos A}$. If $B=A$, $\Sigma=\frac{1}{2}$.

Hence,

$$\cos A-\cos 3A+\cos 5A\ldots\ldots ad\ inf.=\frac{1}{2.\cos A}$$

$$-\cos 2A+\cos 6A-\cos 10A+\quad=-\frac{1}{2.\cos 2A}$$

$$\cos 3A-\cos 9A+\cos 15A\quad=\frac{1}{2.\cos 3A}$$

But by summing the vertical series we have their values, each $\frac{1}{2}$, alternately positive and negative.

$$\therefore \frac{1}{2\cos A} - \frac{1}{2\cos 2A} + \frac{1}{2\cos 3A} - \&c. \ldots ad\ inf. = \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} \ldots ad\ inf.$$

$$\therefore \sec A - \sec 2A + \sec 3A \ldots\ldots\ldots = 1 - 1 + 1 - 1 \ldots\ldots\ldots ad\ inf. = \tfrac{1}{2}$$

Many other inferences might be drawn relative to the series, the sums of which have been obtained. Such the reader cannot fail to deduce for himself; for which reason, without more minutely entering into the detail of them, we shall proceed to other summations.

5. *To sum the series* $\left\{\tan A + \cot A\right\} + \left\{\tan 2A + \cot 2A\right\} + \left\{\tan 4A + \cot 4A\right\}$, &c.

By formula (e) Chap. II. Part I., we have

$$\cot A - \cot 2A = \tfrac{1}{2}\left\{\tan A + \cot A\right\}$$

$$\cot 2A - \cot 4A = \tfrac{1}{2}\left\{\tan 2A + \cot 2A\right\}$$

$$\ldots \quad \ldots \quad \ldots \quad = \quad \ldots \quad \ldots \quad \ldots$$

$$\cot 2^{n-1}.A - \cot 2^{n}.A = \tfrac{1}{2}\left\{\tan 2^{n-1}.A + \cot 2^{n-1}.A\right\}$$

$\therefore$ we have $\cot A - \cot 2^{n}.A = \frac{1}{2}.\Sigma$. Denoting the sum of n terms of the series by the character Σ.

If continued *ad inf.* we have $2\cot A = \Sigma$.

6. *To sum the series* $\frac{1}{2}\tan\frac{1}{2}A + \frac{1}{4}.\tan\frac{1}{4}A + \frac{1}{8}\tan\frac{1}{8} \ldots\ldots \frac{1}{2^{n}}.\tan\frac{1}{2^{n}}A$.

By formula (e) Chap. II. Part I., we have the following formulæ:

$$\tfrac{1}{2}.\cot\tfrac{1}{2}A-\cot A \qquad =\tfrac{1}{2}\tan\tfrac{1}{2}A$$

$$\tfrac{1}{4}.\cot\tfrac{1}{4}A-\tfrac{1}{2}\cot\tfrac{1}{2}A \qquad =\tfrac{1}{4}\tan\tfrac{1}{4}A$$

$$\tfrac{1}{8}.\cot\tfrac{1}{8}A-\tfrac{1}{4}\cot\tfrac{1}{4}A \qquad =\tfrac{1}{8}.\tan\tfrac{1}{8}A$$

$$\ldots \qquad \ldots \qquad \ldots \qquad = \qquad \ldots$$

$$\frac{1}{2^n}.\cot\frac{1}{2^n}A-\frac{1}{2^{n-1}}.\cot\frac{1}{2^{n-1}}.A=\frac{1}{2^n}.\tan\frac{1}{2^n}A$$

$\therefore$ we have $\dfrac{1}{2^n}.\cot\dfrac{1}{2^n}A-\cot A=\Sigma$.

If in this series n be infinite, $\cot\dfrac{1}{2^n}A=1\div\tan\dfrac{1}{2^n}A=\dfrac{2^n}{A}$, $\therefore$ $\Sigma=\dfrac{1}{A}-\cot A$.

Hence $\dfrac{1}{90^\circ}=\tfrac{1}{2}+\tfrac{1}{4}.\tan\tfrac{1}{4}.90^\circ+\tfrac{1}{8}.\tan\tfrac{1}{8}.90^\circ+$ &c.

7. *To sum the series* $\operatorname{cosec} A+\operatorname{cosec} 2A+\operatorname{cosec} 4A$ .... $\operatorname{cosec} 2^{n-1}A$.

By formula (n) Chap. II. Part I., we have the following:

$$\cot\tfrac{1}{2}A-\cot A \qquad =\operatorname{cosec} A$$

$$\cot A-\cot 2A \qquad =\operatorname{cosec} 2A$$

$$\cot 2A-\cot 4A \qquad =\operatorname{cosec} 4A$$

$$\ldots \qquad \ldots \qquad \ldots \qquad = \qquad \ldots$$

$$\cot 2^{n-1}A-\cot 2^{n-1}A=\operatorname{cosec} 2^{n-1}A$$

$\therefore \cot\tfrac{1}{2}A-\cot 2^{n-1}A=\Sigma$.

8. *To sum the series* $\frac{1}{\cos A+\cos 3A}+\frac{1}{\cos A+\cos 5A}+\ldots\ldots\ldots$
$$\frac{1}{\cos A+\cos(2n+1)A}.$$

By formula (32) * Chap. II. Part I., we have the following :]

$$\tan 2A-\tan A \quad =\frac{2.\sin A}{\cos A+\cos 3A}$$

$$\tan 3A-\tan 2A \quad =\frac{2.\sin A}{\cos A+\cos 5A}$$

$$\ldots \quad \ldots \quad = \quad \ldots \quad \ldots$$

$$\tan(n+1)A-\tan nA=\frac{2.\sin A}{\cos A+\cos(2n+1)A}$$

$\therefore \tan(n+1)A-\tan A=2.\sin A.\,\Sigma$, ($\Sigma$ denoting n terms of the series); but

$$\tan(n+1)A-\tan A=\frac{\sin nA}{2.\cos A.\cos(n-1\,A)}\therefore\Sigma=\frac{\sin nA}{\cos A.\cos(n-1\,A}.$$

If the series be continued *ad inf.* $\Sigma=-\frac{1}{2}\sec A$, for the terms subsequent to $\tan A$ continue to destroy.

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* In Chap. II. Part I., formula (32) is not deduced as it should have been from formula (27). Formula (27) may be written

$$\tan A\pm\tan B=\frac{\sin(A\pm B)}{\cos(A+B)+\cos(A-B)},$$

whence we have in general,

$$\tan nA-\tan(n-1)A=\frac{\sin A}{\cos A+\cos(2n-1)A}$$

9. *To sum the series* $\frac{1}{\cos 2A.\cos A}+\frac{1}{\cos 3A.\cos 2A}+\frac{1}{\cos 4A.\cos 3A}$

$$\ldots\ldots\ldots\ldots\frac{1}{\cos(n+1)A.\cos nA}.$$

By formula (27) Chap. II. Part I., we have the following:

$$\tan 2A - \tan A = \frac{\sin A}{\cos 2A.\cos A}$$

$$\tan 3A - \tan 2A = \frac{\sin A}{\cos 3A.\cos 2A}$$

$$\ldots\quad\ldots\quad\ldots = \ldots\quad\ldots$$

$$\tan(n+1)A - \tan nA = \frac{\sin A}{\cos(n+1)A.\cos nA}$$

$$\therefore \tan(n+1)A - \tan A = \sin A.\Sigma$$

or by note in page (88) we have

$$\frac{\sin nA}{\cos nA+\cos(n+2)A} = \sin A.\Sigma,$$

but $\cos nA+\cos(n+2)A = 2.\cos(n+1)A.\cos A$, $\therefore$ we have

$$\Sigma = \frac{\sin nA}{\cos(n+1)A.\sin 2A}$$

10. *To sum the series* $\frac{1}{\sin A.\cos 2A} - \frac{1}{\cos 2A.\sin 3A}$, &c.

From (27) we have, using $(90^\circ - A)$ in place of A, the following formula:

$$\cot A+\tan B = \frac{\cos(A\pm B)}{\sin A.\cos B}.$$

Hence,

$$\cot A + \tan 2A = \frac{\cos A}{\sin A . \cos 2A}$$

$$-\cot 3A - \tan 2A = -\frac{\cos A}{\cos 2A . \sin 3A}$$

$$\ldots \quad \ldots \quad \ldots = \ldots \quad \ldots$$

$$\therefore \cot A - \cot (n+1) A = \cos A . \left\{ \Sigma \text{ to } n \text{ terms, if } n \text{ be even,} \right\}$$

or,

$$\cot A + \tan (n+1) A = \cos A . \left\{ \Sigma \text{ to } n \text{ terms, if } n \text{ be odd.} \right\}$$

11. *To sum the series* $\frac{1}{\sin A . \sin 2A} + \frac{1}{\sin 2A . \sin 3A} \ldots\ldots\ldots\ldots \frac{1}{\sin nA . \sin (n+1) A}$.

By formula (28),

$$\cot A - \cot 2A = \frac{\sin A}{\sin A . \sin 2A}$$

$$\cot 2A - \cot 3A = \frac{\sin A}{\sin 2A . \sin 3A}$$

$$\ldots \quad \ldots \quad \ldots = \ldots \quad \ldots$$

$$\therefore \cot A - \cot (n+1) A = \sin A . \left\{ \Sigma \text{ to } n \text{ terms} \right\}$$

12. *To sum the series* $\frac{1}{\cos A . \sin 2A} - \frac{1}{\sin 2A . \cos 3A} + \&c.$

$$\tan A + \cot 2A = \frac{\cos A}{\cos A . \sin 2A}$$

$$-\tan 3A - \cot 2A = -\frac{\cos A}{\sin 2A . \cos 3A}$$

$$\ldots \quad \ldots \quad \ldots = \ldots \quad \ldots$$

$$\therefore \tan A - \tan (n+1) A = \cos A. \left\{ \Sigma \text{ to } n \text{ terms, if } n \text{ be even} \right\}$$

$$\tan A + \cot (n+1) A = \cos A. \left\{ \Sigma \text{ to } n \text{ terms, if } n \text{ be odd} \right\}$$

13. *To sum the series* $\frac{1}{\sin A. \sin 3A} - \frac{1}{\sin 2A. \sin 4A} + \&c.$

$$\cot A - \cot 3A = \frac{\sin 2A}{\sin A. \sin 3A}$$

$$-\cot 2A + \cot 4A = -\frac{\sin 2A}{\sin 2A. \sin 4A}$$

$$\cot 3A - \cot 5A = \frac{\sin 2A}{\sin 3A. \sin 5A}$$

$$\ldots \quad \ldots = \ldots \quad \ldots$$

$$\cot nA - \cot (n+2) A = -\frac{\sin 2A}{\sin nA. \sin (n+2) A}$$

$$\therefore \cot A - \cot 2A + \cot (n+1) A - \cot (n+2) A = \sin 2A$$
$\left\{ \Sigma. \text{ to } n \text{ terms, if } n \text{ be odd.} \right\}$

$$\cot A - \cot 2A - \cot (n+1) A + \cot (n+2) A = \sin 2A$$
$\left\{ \Sigma. \text{ if } n \text{ be even.} \right\}$

14. *To sum the series* $\frac{1}{\cos A. \cos 3A} - \frac{1}{\cos 2A. \cos 4A} + \frac{1}{\cos 3A. \cos 5A}$

$$\tan 3A - \tan A = \frac{\sin 2A}{\cos A. \cos 3A}$$

$$-\tan 4A + \tan 2A = -\frac{\sin 2A}{\cos 2A. \cos 4A}$$

$$\tan 5A - \tan 3A = \frac{\sin 2A}{\cos 3A. \cos 5A}$$

$\therefore \tan(n+1)A - \tan(n+2)A - (\tan A - \tan 2A) = \sin 2A.$
$\{\Sigma$ to n terms, if n be odd.$\}$

$-\tan(n+1)A + \tan(n+2) - (\tan A - \tan 2A) = \sin 2A.$
$\{\Sigma$ to n terms, if n be even.$\}$

15. *To sum the series* $\cos A . \sin B + \cos 3A . \sin 2B + \cos 5A . \sin 3B + \ldots\ldots\ldots\ldots \cos(2n-1)A . \sin nB.$

$$\sin(A+B) - \sin(A-B) = 2 . \cos A . \sin B$$

$$\sin(3A+2B) - \sin(3A-2B) = 2 . \cos 3A . \sin 2B$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad = \quad \ldots \quad \ldots$$

$$\sin((2n-1)A+nB) - \sin((2n-1)A-nB) = 2 . \cos(2n-1)A . \sin 4B.$$

In adding these we have to find the sums of the vertical columns of the left hand member of the equation. Using θ for $(A+B)$, and ψ for $(2A+B)$, the first of these columns is $\sin\theta + \sin(\theta+\psi) \ldots\ldots\ldots \sin(\theta+(n-1)\psi)$; the sum of which, from Art. (31) of this Chap., is

$$\Sigma = \frac{\sin(\theta + \frac{1}{2}(n-1)\psi)}{\sin\frac{1}{2}\psi} . \sin\frac{n}{2}\psi.$$

Using θ' for $(A-B)$, and ψ' for $(2A-B)$, the sum of the terms of the second column is,

$$\Sigma = \frac{\sin\left(\theta' + \frac{n-1}{2}\psi'\right)}{\sin\frac{1}{2}\psi'} . \sin\frac{n}{2}\psi'.$$

16. *To sum the series* $\frac{1}{2\cos^2\frac{1}{2}A} + \frac{1}{4\cos^2\frac{1}{4}A} + \frac{1}{8.\cos^2\frac{1}{8}A} + \&c.$

$$\frac{1}{\sin A} \qquad - \frac{1}{2\sin^2\frac{1}{2}A} \qquad = \frac{1}{2\cos^2\frac{1}{2}A}$$

$$\frac{1}{2\sin^2\frac{1}{2}A} - \frac{1}{4.\sin^2\frac{1}{4}A} = \frac{1}{4.\cos^2\frac{1}{4}A}$$

$$\ldots \quad \ldots \qquad \ldots \quad \ldots \quad = \quad \ldots \quad \ldots$$

$$\frac{1}{2^{n-1}.\sin^2\frac{1}{2^n}A} - \frac{1}{2^n.\sin^2\frac{1}{2^n}A} = \frac{1}{2^n.\cos^2\frac{1}{2^n}A}$$

$$\therefore \frac{1}{\sin^2 A} - \frac{1}{2^n.\sin^2\frac{1}{2^n}A} = \Sigma$$

Ad inf. $\frac{1}{\sin^2 A} - \frac{1}{A^2} = \Sigma$

17. *To sum the series* $\tan A.\tan^2\frac{1}{2}A + 2\tan\frac{1}{2}A.\tan^2\frac{1}{4}A + \&c.$

$$\tan A - 2\tan\tfrac{1}{2}A = \tan A.\tan^2\tfrac{1}{2}A$$

$$2\tan\tfrac{1}{2}A - 4\tan\tfrac{1}{4}A = 2\tan\tfrac{1}{2}A.\tan^2\tfrac{1}{4}A$$

$$\ldots \quad \ldots \quad \ldots \quad = \quad \ldots \quad \ldots$$

$$\therefore \tan A - 2^n.\tan\frac{1}{2^n} = \Sigma$$

Ad inf. $\tan A - A = \Sigma$

18. *To sum the series* $\frac{1}{\cos^2 A} + \frac{2}{\cos^2 2A} + \frac{4}{\cos^2 4A} = \&c.$

$$\frac{2}{\sin^2 2A} - \frac{1}{\sin^2 A} = \frac{1}{\cos^2 A}$$

$$\frac{4}{\sin^2 4A} - \frac{2}{\sin^2 2A} = \frac{2}{\cos^2 2A}$$

$$\ldots \quad \ldots \qquad \ldots \quad \ldots \quad = \quad \ldots \quad \ldots$$

$$\therefore \frac{2^n}{\sin^2 2^n A} - \frac{1}{\sin^2 A} = \Sigma$$

19. *To sum the series* $\sin A. \sin^2 \tfrac{1}{2} A + 2. \sin \tfrac{1}{2} A. \sin^2 \tfrac{1}{4} A + \&c.$

By formulæ (*a*) and (*b*), Chap. II. Part I., we have the following formulæ:

$$\frac{1}{4}\cdot\left\{2 \sin A - \sin 2A\right\} = \sin A. \sin^2 \tfrac{1}{2} A$$

$$\frac{2}{4}\cdot\left\{2 \sin \tfrac{1}{2} A - \sin A\right\} = 2. \sin \tfrac{1}{2} A. \sin^2 \tfrac{1}{4} A$$

$$\frac{4}{4}\cdot\left\{2 \sin \tfrac{1}{4} A - \sin \tfrac{1}{2} A\right\} = 4. \sin \tfrac{1}{4} A. \sin^2 \tfrac{1}{8} A$$

$$\ldots \quad \ldots \quad \ldots = \ldots \quad \ldots \quad \ldots$$

$$\frac{2^{n-1}}{4}\cdot\left\{2 \sin \frac{1}{2^{n-1}}. A - \sin \frac{1}{2^{n-2}} A\right\} = 2^{n-1}. \sin \frac{1}{2^{n-1}} A. \sin^2 \frac{1}{2^n} A$$

$$\therefore \frac{2^n. \sin \frac{1}{2^{n-1}} A - \sin 2A}{4} = \Sigma. \qquad \textit{Ad inf.}\ \Sigma = \frac{2A - \sin 2A}{4}$$

20. *To sum the series* $\tan A. \sec^2 A + (\tfrac{1}{2} \tan \tfrac{1}{2} A). (\tfrac{1}{2} \sec \tfrac{1}{2} A)^2 + (\tfrac{1}{4} \tan \tfrac{1}{4} A). (\tfrac{1}{4} \sec \tfrac{1}{4} A)^2 + \&c.$

We have

$$\frac{\cos A}{\sin^3 A} - 8. \frac{\cos 2A}{\sin^3 2A} = \tan A \sec^2 A$$

$$\frac{\cos \frac{1}{2} A}{(2 \sin \frac{1}{2} A)^3} - \frac{\cos A}{\sin^3 A} = (\tfrac{1}{2}. \tan \tfrac{1}{2} A). (\tfrac{1}{2} \sec \tfrac{1}{2} A)^2$$

$$\frac{\cos \frac{1}{4} A}{(2^2. \sin \frac{1}{4} A)^3} - \frac{\cos \frac{1}{2} A}{(2. \sin \frac{1}{2} A)^3} = (\tfrac{1}{4}. \tan \tfrac{1}{4} A). (\tfrac{1}{4}. \sec \tfrac{1}{4} A)^2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots = \ldots \quad \ldots \quad \ldots$$

$$\therefore \frac{\cos \frac{1}{2^{n-1}} \cdot A}{\left(2^{n-1} \cdot \sin \frac{1}{2^{n-1}} A\right)^3} - 8 \cdot \frac{\cos 2A}{\sin^3 2A} = \Sigma$$

$$\textit{Ad inf.} \ \frac{1}{A^3} - 8 \cdot \frac{\cos 2A}{\sin^3 2A} = \Sigma$$

21. *To sum the series* $\sin^4 A + 4 \sin^4 \frac{1}{2} A + 4.^2 \sin^4 \frac{1}{4} A + \&c.$

$$\sin^2 A \quad - \tfrac{1}{4} \sin^2 2A = \sin^4 A$$

$$4. \sin^2 \tfrac{1}{2} A - \sin^2 A = 4. \sin^4 \tfrac{1}{2} A$$

$$4^2. \sin^2 \tfrac{1}{4} A - 4. \sin^2 \tfrac{1}{2} A = 4^2. \sin^4 \tfrac{1}{4} A$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots = \ldots \quad \ldots$$

$$\therefore 4^{n-1}. \sin^2 \frac{1}{2^{n-1}} \cdot A - \tfrac{1}{4}. \sin^2 2A = \Sigma$$

Ad inf. $A^2 - \frac{1}{4}. \sin^2 2A = \Sigma.$

22. *To sum the series* $\tan^2 A + (\frac{1}{2} \tan \frac{1}{2} A)^2) + (\frac{1}{4} \tan \frac{1}{4} A)^2 + \&c.$

$$2 + \frac{4}{\tan^2 2A} - \frac{1}{\tan^2 A} = \tan^2 A$$

$$\frac{2}{2^2} + \frac{1}{\tan^2 A} - \frac{1}{(2 \tan \frac{1}{2} A)^2} = (\tfrac{1}{2} \tan \tfrac{1}{2} A)^2$$

$$\frac{2}{2^4} + \frac{1}{(2 \tan \frac{1}{2} A)^2} - \frac{1}{(2^2 \tan \frac{1}{4} A)^2} = \left(\frac{1}{2^2} \cdot \tan \tfrac{1}{4} A\right)^2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = \ldots \quad \ldots$$

$$\frac{2}{2^{2n-2}} + \frac{1}{\left(2^{n-2} \cdot \tan \frac{1}{2^{n-2}} \cdot A\right)^2} - \frac{1}{\left(2^{n-1} \cdot \tan \frac{1}{2^{n-1}} \cdot A\right)^2} = \left(\frac{1}{2^{n-1}} \cdot \tan \frac{1}{2^{n-1}} A\right)^2$$

$$\therefore \left(2.\,2 - \frac{2}{2^{2n-2}}\right) + \frac{4}{\tan^2 2\,A} + \frac{1}{\left(2^{n-1}.\ \tan\frac{1}{2^{n-1}}\,A\right)^2} = \Sigma$$

$$Ad\ inf.\ 4 + \frac{4}{\tan^2 2A} - \frac{1}{A^2} = \Sigma.$$

Several of these series are to be found in the excellent collection of examples on the calculus of finite differences, by Mr. Herchell, and several of them are due to Mr. Wallace, who communicated them to the Royal Society of Edinburgh, in the year 1808. We have them here deduced, perhaps for the first time, by Trigonometrical Artifice. Wallace used some of them as formulæ of approximation to the arc of a circle, (when continued to infinity,) to which purpose their rapid convergence, even in the most unfavourable cases, well adapts them.—*Vide Art.* 6. *of this Chapter*. Mr. Herchell prefers for this purpose the series that have been given in articles (17), (19), (21), as they give the immediate values of the arc and its system, instead of their reciprocals.

It would be wrong not to introduce here some of the very elegant summations that have been performed by Bossut, for an ample detail of which the reader is referred to the *Mem. of the Acad. of Sciences* for the year 1769, page 453.

23. *To sum the series* $\sin^2 A + \sin^2 2\,A + \ldots\ldots\ldots\ldots \sin^2 n\,A$

We have by table (B), Chap. II. Part I. of this treatise,

$$1 - \cos 2\,A = 2.\,\sin^2 A$$

$$1 - \cos 4\,A = 2.\,\sin^2 2\,A$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$1 - \cos 2\,n\,A = 2.\,\sin^2 n\,A$$

Adding together and denoting the sum of $\cos 2\,A + \cos 4\,A + \ldots \ldots\ldots\ldots \cos 2\,n\,A$ by the character Σ, as it can be known from article (1) of this chapter, we have

$$\frac{n}{2}-\tfrac{1}{2}\Sigma=\sin^2 A+\sin^2 2A\ldots\ldots\ldots\ldots\sin^2 nA$$

The sum of this series *ad inf.* is infinite, since Σ in this case is $=-\frac{1}{2}$. This result is somewhat singular, as the same series, when the terms are of the first power, is finite, and $=\frac{1}{2}\cot\frac{A}{2}$. *Art.* (2) *of this Chapter.*

24. *To sum the series* $\cos^2 A+\cos^2 2A+\ldots\ldots\ldots\ldots\cos^2 nA$.

By table (*B*),

$$1+\cos 2A = 2.\cos^2 A$$

$$1+\cos 4A = 2.\cos^2 2A$$

$$\ldots\quad\ldots = \ldots\quad\ldots$$

$$1+\cos 2nA = 2.\cos^2 nA$$

$$\therefore \frac{n}{2}+\tfrac{1}{2}\Sigma=\cos^2 A+\cos^2 2A+\ldots\ldots\ldots\ldots\cos^2 nA.$$

This result might have been obtained from Art. (23), by subtracting both sides of the equation from n.

If the reader use $\frac{\pi}{2}-A$ for A in this and the preceding articles, he will obtain results, differing somewhat in form.

25. *To sum the series* $\sin^3 A+\sin^3 2A\ldots\ldots\ldots\ldots\sin^3 nA$

By table (*A*) Chap. II.

$$3.\sin A-\sin 3A = 4.\sin^3 A$$

$$3.\sin 2A-\sin 6A = 4\sin^3 2A$$

$$\ldots\quad\ldots\quad\ldots = \ldots\quad\ldots$$

$$3.\sin nA-\sin 3nA = 4.\sin^3 nA$$

$\therefore$ denoting $\sin A + \sin 2A \ldots\ldots\ldots\ldots \sin nA$ by Σ, and $\sin 3A + \sin 6A \ldots\ldots\ldots\ldots \sin 3nA$ by Σ', for they can be found by Art. (3), we have

$$3\Sigma - \Sigma' = 4.\left\{\sin^3 A + \sin^3 2A + \sin^3 3A \ldots\ldots\ldots\ldots \sin^3 nA\right\}$$

If continued *ad inf.* this series becomes $\frac{1}{4}\left\{\frac{3}{2}.\cot\frac{A}{2} - \frac{1}{2}\cot\frac{3}{2}A\right\}$ and $\therefore$ finite.

Using for A in this series, $(\pi + A)$, we have the sum of the series $\sin^3 A - \sin^3 2A + \sin^3 3A - \&c.$

26. *To sum the series* $\cos^3 A + \cos^3 2A + \ldots\ldots\ldots\ldots \cos^3 nA$

From table (B), we have

$$3\cos A + \cos 3A \quad = 4.\cos^3 A$$

$$3\cos 2A + \cos 6A = 4.\cos^3 2A$$

$$\ldots \qquad \ldots \qquad = \ldots \qquad \ldots$$

$$3.\cos nA + \cos 3A = 4.\cos^3 nA$$

$\therefore$ denoting $\cos A + \cos 2A$, &c. by Σ, and $\cos 3A + \cos 6A$, &c. by Σ', we have

$$3.\Sigma + \Sigma' = 4.\left\{\cos^3 A + \cos^3 2A + \cos^3 3A \ldots\ldots\ldots\ldots \cos^3 nA\right\}$$

Using for A in this suries, $(\pi - A)$, we can find the sum of the series $\cos^3 A - \cos^3 2A + \cos^3 3A$, &c.

If continued *ad inf.* this series becomes $\frac{1}{4}.\left(-\frac{3}{2} - \frac{1}{2}\right)$ and $\therefore$ finite.

A similar process might be applied to sum $\sin^4 A + \sin^4 2A + \&c.$ and $\cos^4 A + \cos^4 2A$, &c. and all higher powers.

The five preceding series have been taken, with some alterations, from the Memoir by Bossut.

27. *To sum the series* $\sin A + 2.\sin 2A + 3.\sin 3A \ldots\ldots n.\sin nA$, *and* $\cos A + 2\cos 2A + 3.\cos 3A \ldots\ldots n.\cos nA$.

$$\text{Let } S = \sin A + 2.\sin 2A + \ldots\ldots n.\sin nA$$

$$S' = \cos A + 2.\cos 2A + \ldots\ldots n.\cos nA$$

These series may evidently be written as follows:

$$S = \sin A.\left\{1 + 2.\cos A + 3.\cos 2A \ldots\ldots n.\cos(n-1)A\right\}$$

$$+ \cos A.\left\{2.\sin A + 3.\sin 2A + \ldots\ldots n.\sin(n-1)A\right\}$$

$$S' = \cos A.\left\{1 + 2\cos A + 3.\cos 2A \ldots\ldots n.\cos(n-1)A\right\}$$

$$-\sin A.\left\{2.\sin A + 3.\sin 2A \ldots\ldots n.\sin(n-1)A\right\}$$

Hence, (denoting the series $\sin A + \sin 2A \ldots\ldots \sin nA$ by Σ, and $\cos A + \cos 2A \ldots\ldots \cos nA$ by Σ'), we have evidently

$$S = \sin A.\left\{1 + S' - (n+1).\cos nA + \Sigma'\right\} + \cos A.\left\{S - (n+1).\sin nA + \Sigma\right\}$$

$$S' = \cos A.\left\{1 + S' - (n+1).\cos nA + \Sigma'\right\} - \sin A\left\{S - (n+1).\sin nA + \Sigma\right\}$$

Solving these two equations for S and S', we obtain

$$2S = \cot \tfrac{1}{2}A.(1 + \Sigma') - \Sigma + \frac{n+1}{1 - \cos A}.\left\{\sin nA - \sin(n+1)A\right\}$$

$$2S' = -(1 + \Sigma') - \cot \tfrac{1}{2}A.\Sigma + \frac{n+1}{1 - \cos A}.\left\{\cos nA - \sin(n+1)A\right\}$$

These are two series, the summations of which, by any method, have not, I believe, been heretofore published, to a finite number of terms.

The reader will find no difficulty in varying the above series to a great extent, by simply substituting $\left(\frac{\pi}{2} - A\right)$, or $(\pi - A)$, in place of A. We shall forbear entering further on the subject, and proceed to apply one of these already arrived at to the summation of another series.

28. *To sum the series* $1.\cos^2 A + 2.\cos^2 2A + 3.\cos^2 3A \ldots\ldots\ldots\ldots\ldots n.\cos^2 nA$.

$$\tfrac{1}{2}(1+\cos 2A) = \cos^2 A$$

$$\tfrac{1}{2}(2+2.\cos 4A) = 2.\cos^2 2A$$

$$\ldots \quad \ldots \quad \ldots = \ldots \quad \ldots$$

$$\tfrac{1}{2}(n+n.\cos 2nA) = n.\cos^2 nA$$

Adding all these formulæ together, we obtain

$$\frac{n.(n+1)}{4} + \tfrac{1}{2}(\cos 2A + 2\cos 4A \ldots\ldots\ldots\ldots\ldots n.\cos 2nA) = \Sigma,$$

but we have the sum of the series $\cos 2A + 2.\cos 4A \ldots\ldots\ldots\ldots\ldots\ldots n.\cos 2nA$ by Art. (27) of this chapter, $\therefore$ &c.

This series the reader will find at page 52, of Herschell's Examples on the Calculus of finite Differences.

On the subject of this chapter it would be easy, after what has been said, to enlarge to a very great extent. As it may be thought, however, that we have been too diffuse on it already, we shall proceed to matters of more indispensable use, and of more acknowledged utility. Previously to proceeding farther, it will be necessary to become acquainted with the Appendix that we have subjoined on Logarithms.

CHAP. II.

THE THEOREM OF 'DE MOIVRE'—EXPONENTIAL VALUES OF TRIGONOMETRICAL LINES—RELATIONS BETWEEN TRIGONOMETRICAL LINES AND THEIR RESPECTIVE ARCS—EXPRESSIONS FOR SINES AND COSINES OF MULTIPLE ARCS—SERIES FOR THE POWERS OF THE SINE AND COSINE, &c.

1. From the nature of equations, the factors of $\cos^2 A+\sin^2 A$ are $\cos A+\sqrt{-1}.\sin A$, and $\cos A-\sqrt{-1}.\sin A$; hence the equation $\cos^2 A+\sin^2 A=1$, may be written in the form

$$(\cos A+\sqrt{-1}.\sin A).(\cos A-\sqrt{-1}.\sin A)=1;$$

similarly,

$$(\cos B+\sqrt{-1}.\sin B).(\cos B-\sqrt{-1}.\sin B)=1.$$

These factors, though imaginary, are of great use in combining and multiplying arcs.

2. Multiplying together $\cos A\pm\sqrt{-1}.\sin A$ and $\cos B\pm\sqrt{-1}.\sin B$, and substituting $\cos(A+B)$, $\sin(A+B)$, for their developements, we obtain

$$(\cos A\pm\sqrt{-1}.\sin A).(\cos B\pm\sqrt{-1}.\sin B)=\cos(A+B)\pm\sqrt{-1}.\sin(A+B).$$

If $B=A$, this Theorem becomes

$$(\cos A\pm\sqrt{-1}.\sin A)^2=\cos 2A\pm\sqrt{-1}.\sin 2A.$$

If this be true of n arcs it is true of $(n+1)$ arcs, for in the equa-

tion $(\cos A \pm \sqrt{-1}.\sin A)^n = \cos nA \pm \sqrt{-1}.\sin nA$, let both sides be multiplied by $\cos A \pm \sqrt{-1}.\sin A$, and we shall obtain

$$(\cos A \pm \sqrt{-1}.\sin A)^{n+1} = \cos(n+1)A \pm \sqrt{-1}.\sin(n+1)A.$$

The Theorem being true for two arcs, is, by this argument, true for n arcs, n being a positive integer.

The same is true for a positive fractional index, $\frac{m}{n}$, for by the above $(\cos A \pm \sqrt{-1}.\sin A)^m = \cos mA \pm \sqrt{-1}.\sin mA$, but $\cos A \pm \sqrt{-1}.\sin A = (\cos nA \pm \sqrt{-1}.\sin nA)^{\frac{1}{n}}$ $\therefore$ we have $(\cos nA \pm \sqrt{-1}.\sin nA)^{\frac{m}{n}} = \cos mA \pm \sqrt{-1}.\sin mA$. For nA in this last equation write A', and it becomes

$$(\cos A' \pm \sqrt{-1}.\sin A')^{\frac{m}{n}} = \cos\frac{m}{n}A' \pm \sqrt{-1}.\sin\frac{m}{n}A';$$

or, restoring the former notation,

$$(\cos A \pm \sqrt{-1}.\sin A.)^{\frac{m}{n}} = \cos\frac{m}{n}A \pm \sqrt{-1}.\sin\frac{m}{n}A.$$

The same is true for any negative index, for

$$(\cos A \pm \sqrt{-1}.\sin A)^{-m} = \frac{1}{\cos mA \pm \sqrt{-1}.\sin mA};$$

multiply the right hand member of this equation, above and below, by $\cos mA \pm \sqrt{-1}.\sin mA$, and we obtain

$$(\cos A \pm \sqrt{-1}.\sin A)^{-m} = \cos mA \mp \sqrt{-1}.\sin mA.$$

This is the celebrated* Theorem of De Moivre, which he ar-

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* "*Formule remarquable autant par sa simplicite et son elegance que par sa generalité et sa fecondité.*"—Lagrange *leçons sur le Calcul des Fonctions*, page 116.

rived at by the comparison of hyperbolic with circular sectors. The proof above given of it is founded on the inductive process alluded to in Chap. II. Part I. of this work, and is, perhaps, for the first time extended to negative and fractional indices. Lacroix, who divides it into two Theorems, after proving it for indices that are *positive integers*, and without verifying the induction, makes the following remark: "*Mais allors elles ne seront prouvees que pour le cas ou le nombre* n *est entier.*"

3. From the formula of De Moivre we can obtain formulæ for the sines, cosines, and tangents of multiple arcs.

$$\sin nA = \frac{(\cos A + \sqrt{-1}.\sin A)^n - (\cos A - \sqrt{-1}.\sin A)^n}{2\sqrt{-1}}$$

$$\cos nA = \frac{(\cos A + \sqrt{-1}.\sin A)^n + (\cos A - \sqrt{-1}.\sin A)^n}{2}$$

By expanding and arranging, we have from these

$$\left.\begin{array}{l} \sin nA = n.\sin A.\cos^{n-1} A - \dfrac{n.n-1.n-2}{1.2.3.}.\sin^3 A.\cos^{n-3} A \ldots\ldots \\ \dfrac{n.n-1\ldots\ldots\ldots\ldots\left\{n-(2m-2)\right\}}{1.2.3\ldots\ldots\ldots\ldots\ldots\ldots(2m-1)}.\sin^{2m-1} A.\cos^{n-(2m-1)} A \ldots \end{array}\right\} (1)$$

$$\left.\begin{array}{l} \cos nA = \cos^n A - \dfrac{n.n-1}{1.2}.\cos^{n-2} A.\sin^2 A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{n.n-1\ldots\ldots\ldots\ldots\left\{n-(2m-3)\right\}}{1.2\ldots\ldots\ldots\ldots\ldots\ldots(2m-2)}.\cos^{n-(2m-2)} A.\sin^{2m-2}\ldots\ldots \end{array}\right\} (2)$$

These series will terminate, only when n is a positive integer.

Dividing these expressions one by the other, and then dividing the numerator and denominator of the right hand member by $\cos^n A$, we obtain series (σ), Chap. II. Part I.

The formulæ (1) and (2) were given by John Bernouilli, in the

Leipsic Acts for 1701, without demonstration. It would appear from the *Commercium Epist.* letter 129, that they were found by induction. It was not until twenty years after, however, that finite expressions for sin nA and cos nA were arrived at by De Moivre. "Thus," says La Grange, "John Bernouilli touched twice* upon the same discovery, and still left the glory of it to his successors."

4. Let $nA = A'$, where A' is a finite arc. As n becomes great, A must diminish, and this increase and diminution being supposed to go on *sine limite*; $(n-1)$, $(n-2)$, &c. tend each to become n, cos A tends to unity as its limit, and sin A tends to become A. Using then $\frac{A'}{n}$ for A, after these changes have happened, the two series, (1) and (2), become

$$\sin A' = A' - \frac{A'^3}{1.2.3} + \frac{A'^5}{1.2.3.4.5} - \&c.$$

$$\cos A' = 1 - \frac{A'^2}{1.2} + \frac{A'^4}{1.2.3.4} - \&c.$$

Or restoring, for uniformity sake, the old notation, we have

$$\sin A = A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \&c. \qquad (3)$$

$$\cos A = 1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \&c. \qquad (4)$$

5. In the developement of e^x, (*Vide Appendix,*) substituting $A.\sqrt{-1}$, $-A.\sqrt{-1}$, for x, we obtain

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* Here La Grange alludes to a paper of Bernouilli's, in the *Mem. of the Academy* for 1702.

$$e^{A\sqrt{-1}}=1+A.\sqrt{-1}-\frac{A^2}{1.2}-\frac{A^3.\sqrt{-1}}{1.2.3}+\frac{A^4}{1.2.3.4}+\&c.$$

$$e^{-A\sqrt{-1}}=1-A\sqrt{-1}-\frac{A^2}{1.2}+\frac{A^3\sqrt{-1}}{1.2.3}+\frac{A^4}{1.2.3.4}-\&c.$$

Adding and subtracting these, and by series (3) and (4), we obtain the following singular expressions for the sine and cosine,

$$\sin A = \frac{e^{A\sqrt{-1}}-e^{-A\sqrt{-1}}}{2.\sqrt{-1}}; \cos A=\frac{e^{A\sqrt{-1}}+e^{-A\sqrt{-1}}}{2} \qquad (5)$$

Hence we easily have

$$\tan A = \frac{1}{\sqrt{-1}}\cdot\frac{e^{2A\sqrt{-1}}-1}{e^{2A\sqrt{-1}}+1}$$

Thus we find sines and cosines expressed by imaginary exponentials, " which," says La Grange, " we ought to regard as one of the happiest analytical discoveries that have been made in this age."—*Calcul des Fonctions*, page 114.

These expressions are due to John Bernoulli, who gives them in a few words in the *Mem. of the Acad.* for 1702.

6. The manner here given of arriving at these expressions, along with being exceedingly simple, possesses the advantage of shewing us the true meaning of such formulæ, it proves that they are no other than purely Algebraic symbols, by which we express, in brief, a set of operations or developements to be effected for arriving at a sine or cosine. They by no means express real values, for the terms $e^{A\sqrt{-1}}$, $e^{-A\sqrt{-1}}$, are no other than *analogical* expressions formed on the model of e^A and e^{-A}, by substituting $A\sqrt{-1}$ for A, and which, not having of themselves any value, cannot be conceived or interpreted unless by their developement.

7. The reader will doubtless observe, that the proof given of De Moivre's Theorem, at the beginning of this chapter, applies only to the case where the index is *real* and *rational*. We can now complete the proof for *surd* and *imaginary* values.

By formula (5) we have $e^{A\sqrt{-1}}=\cos A+\sqrt{-1}.\sin A$, where A is any arc, $\therefore e^{mA\sqrt{-1}}=\cos mA+\sqrt{-1}.\sin mA=(\cos A+\sqrt{-1}.\sin A)^m$. Similarly,

$$e^{-mA\sqrt{-1}}=\cos mA-\sqrt{-1}.\sin mA=(\cos A-\sqrt{-1}.\sin A)^m$$

Hence the Theorem is universal. Thus, perhaps, we have supplied the deficiency observed by La Grange, who remarks, that the Theorem cannot be proved in its full generality without the consideration of derivative functions.—*Calcul des Fonctions*, page 117.

8. Dividing one by the other the value of $e^{A.\sqrt{-1}}$ and $e^{-A.\sqrt{-1}}$, we obtain

$$e^{2A\sqrt{-1}}=\frac{1+\sqrt{-1}.\tan A}{1-\sqrt{-1}.\tan A}\ \therefore 2A.\sqrt{-1}=\log.\frac{1+\sqrt{-1}.\tan A}{1-\sqrt{-1}.\tan A};$$

but

$$\log.\frac{1+x}{1-x}=2.\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\&c.\right\},$$

$\therefore$ if x be replaced by $\sqrt{-1}.\tan A$, we have

$$A=\tan A-\frac{\tan^3 A}{3}+\frac{\tan^5 A}{5}-\frac{\tan^7 A}{7}+\&c. \qquad (6)$$

This beautiful and simple series was first arrived at by Leibnitz, by integrating the differential of the arc in terms of the tangent.

9. Having thus found a series for the arc in terms of the tangent, we shall now proceed to deduce the same in terms of the other Trigonometrical lines. First, for the sine, assume

$$A=\beta.\sin A+\gamma.\sin^3 A+\delta.\sin^5 A+\&c. \qquad (m)$$

when β, γ, δ, &c. are indeterminate co-efficients. Then we have two forms for $2A$ from (m), viz.

$$\left.\begin{array}{l}2\,A = 2\,\beta.\sin A.\cos A + 8.\,\gamma.\sin^3 A.\cos^3 A + 32.\,\delta.\sin^5 A.\\ \qquad\cos^5 A + \&c.\end{array}\right\}(n)$$

$$2\,A = 2\,\beta.\sin A + 2\,\gamma.\sin^3 A + 2\,\delta.\sin^5 A + \&c. \qquad (o)$$

Arranging series (n) by the powers of $\sin A$, we have

$$\left.\begin{array}{l}2\,A = 2\,\beta.\sin A - (\beta - 8.\,\gamma).\sin^3 A - (\tfrac{1}{4}.\,\beta + 12.\,\gamma - 32.\,\delta + 126.\,\epsilon)\\ \qquad\sin^5 A + \&c.\end{array}\right\}(p)$$

In the series (o) and (p), equating the co-efficients of like powers of $\sin A$, we obtain

$$\beta = \beta,\ \gamma = \frac{\beta}{2.\,3},\ \delta = \frac{3.\,\beta}{2.\,4.\,5},\ \epsilon = \frac{3.\,5.\,\beta}{2.\,4.\,6.\,7},\ \&c.$$

whence we have

$$A = \beta.\left\{\sin A + \frac{\sin^3 A}{2.\,3} + \frac{3.\sin^5 A}{2.\,4.\,5}\ldots\ldots\frac{3.\,5\ldots\ldots(2n-3)\sin^{2n-1} A}{2.4.6\ldots(2n-2).\,(2n-1)}.\right\}(7)$$

In this series the limit of the ratio $\sin A$ and A, is β; but in general this limit is unity, $\therefore\ \beta = 1$.

10. Using for A in this series, $\frac{1}{2}\pi - A$, we obtain the developement of A in a series ascending by the powers of the cosine,

$$A = \tfrac{1}{2}\pi - \left\{\cos A + \frac{\cos^3 A}{2.\,3} + \frac{3.\cos^5 A}{2.\,4.\,5}\ldots\frac{3.5\ldots\ldots(2n-3)\cos^{2n-1} A}{2.4.6\ldots(2n-2).(2n-1)}.\right\}(8)$$

11. From series (6), by a similar substitution, we obtain a series for A in terms of $\cot A$, viz.

$$A = \tfrac{1}{2}\pi - \left\{\cot A - \tfrac{1}{3}.\cot^3 A + \tfrac{1}{5}.\cot^5 A - \&c.\right\} \qquad (9)$$

12. From series (3), using $\frac{1}{2}A$ for A, we obtain a series for the difference between the arc and chord,

$$A-2.\sin\tfrac{1}{2}A=\frac{A^3}{1.2.3.2^2}-\frac{A^5}{1.2.3.4.5.2^4}+\frac{A^7}{1.2.3.4.5.6.7.2^6}-\&c. \quad (10)$$

For the same, we have from (7) the following series:

$$A-2.\sin\tfrac{1}{2}A=\frac{\sin^3\frac{1}{2}A}{3}+\frac{3.\sin^5\frac{1}{2}A}{4.5}+\frac{3.5.\sin^7\frac{1}{2}A}{4.6.7}+\&c. \quad (11)$$

Hence we can always compute the chords to any degree of exactness. This was the method used by the early writers on Trigonometry.

13. The expression for $\tan A$ in terms of A, may be deduced by several methods. The easiest, and perhaps the most simple, is, by dividing series (3) by (4). Hence we obtain the following series:

$$\tan A=A+\frac{A^3}{3}+\frac{2.A^5}{3.5}+\frac{17.A^7}{3^2.5.7}+\frac{62.A^9}{3^2.5.7.9}+\frac{1382.A^{11}}{3^2.5^2.7.9.11}+\frac{21844.A^{13}}{3^2.5^2.7.9.11.13}+\frac{929569.A^{15}}{3^2.5^2.7^2.9.11.13.15}+\&c.$$

or the following more convenient one,

$$\left.\begin{array}{l}\tan A=A+\dfrac{2.A^3}{1.2.3}+\dfrac{16.A^5}{1.2.3.4.5}+\dfrac{272.A^7}{1.2.3.4.5.6.7}+\\ \dfrac{8936.A^9}{1.2.3.4.5.6.7.8.9}+\&c.\end{array}\right\}(12)$$

This series may be conveniently found as follows. From series (3) and (4) we find by clearing of fractions and arranging,

$$\tan A=A+\frac{A^2.\tan A}{1.2}-\frac{A^3}{1.2.3}-\frac{A^4\tan A}{1.2.3.4}+\&c.$$

for $\tan A$, in each term of this expansion substitute

$$A+\frac{A^2.\tan A}{1.2}-\&c.$$

in the result obtained, substitute again for tan A the same expression, arrange the terms according to the powers of A, and series (12) is the result.

14. Dividing unity by series (4), we find for cosec A

$$\text{cosec}\, A = \frac{1}{A} + \frac{A}{1.\,2.\,3} + \frac{14.\,A^3}{1.\,2.\,3.\,4.\,5.\,6} + \frac{744.\,A^5}{1.\,2.\,3.\,4.\,5.\,6.\,7.\,8.\,9} + \&c. \quad (13)$$

15. Dividing unity by series (12), or dividing (5) by (4), we find

$$\cot A = \frac{1}{A} - \frac{2\,A}{1.\,2.\,3} - \frac{32.\,A^3}{1.\,2.\,3.\,4.\,5.\,6} - \frac{768.\,A^5}{1.\,2.\,3.\,4.\,5.\,6.\,7.\,8.\,9} - \&c. \quad (14)$$

16. Dividing unity by series (5), we find

$$\sec A = 1 + \frac{A^2}{1.\,2} + \frac{5.\,A^4}{1.\,2.\,3.\,4} + \frac{61.\,A^6}{1.\,2.\,3.\,4.\,5.\,6} + \&c. \quad (15)$$

This series was first given by James Gregory.—*Vide Com. Epistolicum.*

17. By series (k), in the Appendix, we shall be enabled to find an expression for the arc of a circle, arranged in terms of the sines of its multiples. If in

$$l.\, u = (u - u^{-1}) - \frac{(u^2 - u^{-2})}{2} + \frac{u^3 - u^{-3}}{3} - \frac{u^4 - u^{-4}}{4} + \&c.$$

we make $u = e^{A\sqrt{-1}}$, and $\therefore l.\, u = A\sqrt{-1}$, the series, becomes

$$A = \frac{e^{A\sqrt{-1}} - e^{-A\sqrt{-1}}}{\sqrt{-1}} - \tfrac{1}{2}.\, \frac{e^{2A\sqrt{-1}} - e^{-2A\sqrt{-1}}}{\sqrt{-1}} + \tfrac{1}{3}.\, \frac{e^{3A\sqrt{-1}} - e^{-3A\sqrt{-1}}}{\sqrt{-1}}$$

or, dividing each term by 2, and substituting the sines for their logarithmic values, we find

$$\tfrac{1}{2}\, A = \sin A - \tfrac{1}{2}.\, \sin 2\, A + \tfrac{1}{3}.\, \sin 3\, A - \tfrac{1}{4}.\, \sin 4\, A + \&c. \quad (16)$$

18. Another very singular expression may be deduced as follows:

$$\sin A = 2.\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}A = 4\sin\tfrac{1}{4}A.\cos\tfrac{1}{4}A.\cos\tfrac{1}{2}A = 8.\cos\tfrac{1}{2}A.\cos\tfrac{1}{4}A.\cos\tfrac{1}{8}A.\sin\tfrac{1}{8}A,\ \&c.$$

all from the formula

$$\sin A = 2.\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}A$$

as this lengthening out of the series may be continued to any extent, we have ultimately

$$\sin A = 2^n.\sin\frac{A}{2^n}.\cos\tfrac{1}{2}A.\cos\tfrac{1}{4}A.\cos\tfrac{1}{8}A \ldots\ldots\ldots\ldots \cos\frac{1}{2^n}A.$$

In this, if n become infinitely great,

$$\sin\frac{1}{2^n}A = \frac{1}{2^n}.A$$

by using which, and arranging accordingly, we find

$$A = \sin A.\sec\tfrac{1}{2}A.\sec\tfrac{1}{4}A.\sec\tfrac{1}{8}A.\ \&c.\ \textit{ad inf.} \qquad (17)$$

19. Series (16) may be converted into another very elegant series by writing for A, $\pi - A$, by which it becomes

$$\frac{\pi}{2} - \frac{A}{2} = \sin A + \tfrac{1}{2}.\sin 2A + \tfrac{1}{3}.\sin 3A + \tfrac{1}{4}\sin 4A + \&c. \qquad (18)$$

20. Adding them, we obtain a beautiful series for $\frac{\pi}{4}$, viz.

$$\frac{\pi}{4} = \sin A + \tfrac{1}{3}.\sin 3A + \tfrac{1}{5}\sin 5A + \&c. \qquad (19)$$

These two latter series are given by Euler, *Inst. Calc. Diff.* and thence quoted by Peacock in his Examples on the Calculus. They are here deduced by a method different from either, and much simpler.

21. From the imaginary value for the tangent already found, we have

$$\sqrt{-1}.\tan x = -\frac{1-e^{2x\sqrt{-1}}}{1+e^{2x\sqrt{-1}}} = -(1-e^{2x\sqrt{-1}})(1+e^{2x\sqrt{-1}})^{-1}$$

which, being expanded, gives

$$\sqrt{-1}.\tan x = -1+2.\left\{e^{2x\sqrt{-1}}-e^{4x\sqrt{-1}}+e^{6x\sqrt{-1}}-\&c.\ ad\ inf.\right\}$$

but

$$e^{2x\sqrt{-1}} = \cos 2x + \sqrt{-1}.\sin 2x,\ \&c.$$

∴ by substituting such expressions, we have

$$\sqrt{-1}.\tan x = -1+2(\cos 2x - \cos 4x + \cos 6x - \&c.)$$

$$+2\sqrt{-1}.(\sin 2x - \sin 4x + \sin 6x - \&c.)$$

but in such an equation, *vide Wood's Algebra*, page 127, the real and imaginary parts must separately destroy, whence we have, using A for $2x$,

$$\cos A - \cos 2A + \cos 3A - \&c.\ ad\ inf. = \tfrac{1}{2} \qquad (20)$$

and

$$\sin A - \sin 2A + \sin 3A - \&c.\ ad\ inf. = \tfrac{1}{2}\tan\tfrac{1}{2}A \qquad (21)$$

These two series the reader may see differently obtained in Chap. I. Part II. of this treatise.

22. From the logarithmic expression for the cosine, we have

$$\tfrac{1}{2}\sec x = \frac{e^{x\sqrt{-1}}}{e^{2x\sqrt{-1}}+1} = e^{x\sqrt{-1}}\left\{1-e^{2x\sqrt{-1}}+e^{4x\sqrt{-1}}-e^{6x\sqrt{-1}}+\&c.\right\}$$

$$=(\cos x+\sqrt{-1}.\sin x)\left\{\begin{array}{l}1-\cos 2x+\cos 4x-\cos 6x+\&c.\\ -\sqrt{-1}(\sin 2x-\sin 4x+\&c.)\end{array}\right\}$$

In which, if the real and imaginary parts be separated, and $\cos 3x$, $\cos 5x$, &c. $\sin 3x$, $\sin 5x$, &c. substituted for their equivalents, we obtain values already obtained in Chapter I. Part II. of this treatise.

23. From the expression for the arc in terms of the tangent given in series (6) of this Chapter, we have

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{\&c.}$$

This expression may be converted into a continued fraction, as follows:

$$\text{Assume } \frac{\pi}{4} = \cfrac{a}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \text{\&c.}}}}}$$

She several successive values of the fraction are

$$\frac{a}{b};\ \frac{ac}{bc+\beta};\ \frac{a.(dc+\gamma)}{bcd+b\gamma+\beta d};$$

$\therefore$ the differences of the first and second of these values, is

$$\frac{a\beta}{b.(bc+\beta)};$$

of the second and third,

$$\frac{a\beta\gamma}{(bc+\beta).(bcd+\beta d+\gamma b)};\ \text{\&c.}$$

Hence we have

$$\frac{\pi}{4} = \frac{a}{b} - \frac{a\beta}{b.(bc+\beta)} + \frac{a\beta\gamma}{(bc+\beta).(bcd+\beta d+\gamma b)} - \text{\&c.};$$

now that this should be identical with the above, we must have $a = b$; $\beta = \frac{1}{2}.\, bc$; $\gamma = \frac{9}{4}.\, cd$; $\delta = \frac{25}{4}.\, de$, &c. in which b, c, d, &c. remain arbitrary. Assume $b = 1$; $c = 2$; $d = 2$; $e = 2$; &c. then we have

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{\&c.}}}}}$$

This singular expression is due to Lord Brouncker. It is to be found in Wallis's works, but deduced by a method different from the above.

24. The values given for the sine and cosine, in an exponential form, may be exhibited in a figure as follows. Take a semicircle, $pc''o$, (Fig. 17.) with a radius unity, take in it an arc, $oc = 2A$, draw the chord pc, then $pc = 2\sin\left(\frac{180 - 2A}{2}\right) = 2\cos A$; take in succession $oc_{,}$, $oc_{,,}$ &c. $= 4A$, $6A$. &c. then $pc_{,} = 2\cos 2A$, $pc_{,,} = 2\cos 3A$, &c. Denoting the exponential value of $2\cos A$ by $x + \frac{1}{x}$, then $pc = x + \frac{1}{x}$, $pc_{,} = x^2 + \frac{1}{x^2}$, $pc_{,,} = x^3 + \frac{1}{x^3}$, &c. .... $pc_{n-1} = x^n + \frac{1}{x^n}$. This is obvious from the exponential values, for $2\cos A = e^{A\sqrt{-1}} + e^{-A\sqrt{-1}}$, and for the same reason $2\cos 2A = e^{2A\sqrt{-1}} + e^{-2A\sqrt{-1}}$, viz. the sum of the squares of the former values, &c.

25. The roots of the equation $z^2 - pz + 1 = 0$ are of the form x and $\frac{1}{x}$, hence if pc be taken to represent their sum, $pc_{,}$ represents the sum of their squares, $pc_{,,}$ the sum of their cubes, &c.

Thus the Theorem is given by *Waring*, in his properties of curves; and implicitly in the same form by the inventor *Vieta*.

The chord $oc = 2 \sin A = \frac{1}{\sqrt{-1}} \cdot \left(x - \frac{1}{x}\right) \therefore oc_{,} = \frac{1}{\sqrt{-1}} \cdot \left(x^2 - \frac{1}{x^2}\right)$, $oc_{,,} = \frac{1}{\sqrt{-1}} \cdot \left(x^3 - \frac{1}{x^3}\right)$, &c. $oc_{n-1} = \frac{1}{\sqrt{-1}} \cdot \left(x^n - \frac{1}{x^n}\right)$. This is obvious also from the exponential value for the sine.

Let us recollect the Algebraic Theorem, that *the difference of the* n^{th} *powers of any two quantities, divided by their difference, is equal to their sum in the power* (n—1), *wanting its co-efficients.* Denoting then $\frac{1}{\sqrt{-1}}\left(x - \frac{1}{x}\right)$ by a, and dividing the successive terms by it, we have

$$oc_{,} = a\left(x + \frac{1}{x}\right),\ oc_{,,} = a\left(x^2 + 1 + \frac{1}{x^2}\right),\ oc_{,,,} = a\left(x^3 + x + \frac{1}{x} + \frac{1}{x^3}\right), \text{ \&c.}$$

expressions the same in substance as those given by Vieta for the same lines when $a = 1$.

The triangles ocp, $oc_{,}p$, $oc_{,,}p$, &c. are represented by the quantities

$$\frac{1}{\sqrt{-1}}\left(x^2 - \frac{1}{x^2}\right),\ \frac{1}{\sqrt{-1}}\left(x^4 - \frac{1}{x^4}\right),\ \frac{1}{\sqrt{-1}}\left(x^6 - \frac{1}{x^6}\right), \text{ \&c.}$$

but the triangles are also as the perpendiculars cn, $c_{,}n_{,}$, $c_{,,}n_{,,}$, $c_{,,,}n_{,,,}$, &c. ∴ the perpendiculars are represented by these values.

The versed sines on, $on_{,}$, $on_{,,}$, &c. are denoted by the quantities

$$\left(x - \frac{1}{x}\right)^2,\ \left(x^2 - \frac{1}{x^2}\right)^2,\ \left(x^3 - \frac{1}{x^3}\right)^2, \text{ \&c.}$$

as appears by considering that $pc : co :: cn : no$.

26. The exponential values for the cosine are used by Woodhouse in his Trigonometry to sum the series

$$\cos A + \cos 2A + \cos 3A \ldots\ldots\ldots\ldots \cos nA$$

by summing the two Geometric progressions,

$$\tfrac{1}{2}\left\{x + x^2 + x^3 \ldots\ldots\ldots\ldots x^n\right\}$$

and

$$\tfrac{1}{2}\left\{\frac{1}{x} + \frac{1}{x^2} + \&c. \ldots\ldots\ldots\ldots \frac{1}{x^n}\right\}.$$

A similar method might be used for the series

$$\sin A + \sin 2A \ldots\ldots\ldots\ldots \sin nA,$$

by summing the series

$$\frac{1}{2\sqrt{-1}}\left\{x + x^2 + \ldots\ldots\ldots\ldots x^n\right\}$$

and

$$\frac{1}{2\sqrt{-1}}\left\{\frac{1}{x} + \frac{1}{x^2} + \ldots\ldots\ldots\ldots \frac{1}{x^n}\right\}.$$

Woodhouse fears, however, to apply it to this latter series, lest he might introduce *imaginary* symbols. This objection will doubtless seem ludicrous to the student who knows the true meaning of the quantities x and $\frac{1}{x}$. The method for both series is decidedly inferior to that given for more general ones in the first chapter of the second part of this treatise.

27. Two very important questions remain still to be discussed in this Chapter;—1st, To deduce expressions for the sines and cosines of multiple arcs in series of powers of the sines and cosines of the simple arcs; and 2dly, Expressions for the n^{th} powers of the sines and cosines in terms of the sines and cosines of the multiples.

The developements given in page 103 of this work would completely resolve the first of these problems, if our object were merely to obtain values for the sine and cosine of a multiple arc, by means of the powers of the sine and cosine of a simple arc, their only inconvenience, (an inconvenience resulting from the nature of the case,) is that they would not terminate if the index became fractional or negative. They contain, however, at the same time, powers both of the sine and cosine which seems an useless complication, and one ∴ that ought to be avoided.

From the Theorem of De Moivre, we shall proceed to deduce series *a priori*, to remedy this inconvenience. We have from it,

$$2.\cos nx = (\cos x + \sqrt{-1}.\ \sin x)^n + (\cos x - \sqrt{-1}.\ \sin x)^n$$

$$2.\sin nx = \left\{(\cos x + \sqrt{-1}.\sin x)^n - (\cos x - \sqrt{-1}.\ \sin x)^n\right\}\frac{1}{\sqrt{-1}}$$

Substituting p for $\cos x$, and q for $\sin x$, the former becomes

$$2.\ \cos nx = (p + \sqrt{p^2 - 1})^n + (p - \sqrt{p^2 - 1})^n$$

which we shall first proceed to discuss.

Let

$$(p + \sqrt{p^2 - 1})^n = Ap^n + Bp^{n-1} + Cp^{n-2} + \&c.;$$

for p substitute $\frac{1}{z}$, then multiplying both sides of this equation by z^n, we have

$$(1 + \sqrt{1 - z^2})^n = A + Bz + Cz^2 + \&c.;$$

expanding by the binomial, the first thing obvious is, that there are no odd powers of z, and ∴ that $B=0$, $D=0$, &c. next

$$1 + n + \frac{n.\ n-1}{1.\ 2} + \frac{n.\ n-1.\ n-2}{1.\ 2.\ 3},\ \&c. = (1+1)^n = 2^n = A$$

$$-\frac{n}{2}\left(1 + \frac{n-1}{1} + \frac{n-1}{1}.\ \frac{n-2}{2} + \&c.\right) = -n.\ 2^{n-2} = C$$

$$+\frac{n.(n-3)}{4}\left(1+\frac{n-3}{1}+\frac{n-3}{1}.\ \frac{n-4}{2}+\&c.\right)=\frac{n.\,n-3}{2}.\ 2^{n-4}=E$$

$$-\frac{n.(n-4).(n-5)}{4.\,3}\left(1+\frac{n-5}{1}+\frac{n-5}{2}.\frac{n-6}{2}+\&c.\right)=-$$
$$\frac{n.\,n-4.\,n-5}{2.\,3}.\ 2^{n-6}=G$$

$\therefore$ restoring p for $\frac{1}{z}$, and arranging the terms, we find

$$(p+\sqrt{p^2-1})^n=(2p)^n-\frac{n}{1}.(2p)^{n-2}+\frac{n.\,n-3}{1.2}.(2p)^{n-4}$$
$$+n.\frac{n-4.\,n-5}{1.\,2.\,3}.(2p)^{n-6}+\&c.$$

It still remains to expand

$$(p-\sqrt{p^2-1})^n\,;$$

now it is obvious that

$$p-\sqrt{p^2-1}=\frac{1}{p+\sqrt{p^2-1}}\,;\ \therefore (p-\sqrt{p^2-1})^n=(p+\sqrt{p^2-1})^{-n}$$

the expansion of which is had from the preceding, by merely changing the sign of n; $\therefore$ the complete developement of the $\cos nx$ is expressed in a compound series, as follows:

$$\left.\begin{array}{l}2.\cos nx=(2p)^n-n.(2p)^{n-2}+\frac{n.\,n-3}{1.\,2}.(2p)^{n-4}-\&c.\\ \qquad (2p)^{-n}+n\,(2p)^{-n-2}+\frac{n.(n+3)}{1.\,2}.(2p)^{-n-4}+\&c.\end{array}\right\}(1)$$

If for x, we use $\left(\frac{\pi}{2}-x\right)$, we obtain from this series the two following:

1° If n be odd,

$$\left.\begin{array}{l}\pm 2.\sin nx=(2q)^n-n.(2q)^{n-2}+\frac{n.\overline{n-3}}{1.2}.(2q)^{n-4}-\&c.\\ \quad +(2q)^{-n}+n(2q)^{-n-2}+\&c.\end{array}\right\}(2)$$

2° If n be even,

$$\left.\begin{array}{l}\pm 2.\cos nx=(2q)^n-n.(2q)^{n-2}+\frac{n.\overline{n-3}}{1.2}.(2q)^{n-4}-\&c.\\ \quad +(2q)^{-n}+n(2q)^{-n-2}+\frac{n.\overline{n+3}}{1.2}.(2q)^{-n-4}=\&c.\end{array}\right\}(3)$$

The value of $2.\sin nx$ given by De Moivre's Theorem, if treated as the above, may be developed in a series descending by the powers of the cosine. Such a mode of proceeding would be nugatory and useless, when the same developement can be obtained by taking the differential of both sides of equation (1). The result thus obtained being divided by $n\,dx$, gives

$$\left.\begin{array}{l}\sin nx=q\left\{(2p)^{n-1}-(n-2)(2p)^{n-3}+\frac{\overline{n-3}.\overline{n-4}}{1.2}.(2p)^{n-5}-\&c.\right\}\\ \quad -q\left\{(2p)^{-n-1}+(n+2)(2p)^{-n-3}+\&c.\right\}\end{array}\right\}(4)$$

In this substitute $\frac{\pi}{2}-x$ for x, and we obtain,

1° If n be odd,

$$\left.\begin{array}{l}\pm\cos nx=p\left\{(2q)^{n-1}-(n-2)(2q)^{n-3}+\&c.\right\}\\ \quad -p\left\{(2q)^{-n-1}+(n+2)(2q)^{-n-3}+\&c.\right\}\end{array}\right\}(5)$$

2° If n be even,

$$\left.\begin{aligned}\pm \sin n x = p &\left\{(2q)^{n-1} - (n-2)(2q)^{n-3} + \&c.\right\}\\ -p &\left\{(2q)^{-n-1} - (n-2)(2q)^{-n-3} + \&c.\right\}\end{aligned}\right\} \quad (6)$$

It has been usual to deduce the series here given by the aid of an higher calculus. Perhaps the complete developements have not heretofore been effected by elementary Algebra.

If the reader refer to the methods of deducing them in Lagrange, *Leçons sur le Calcul des Fonctions*, page 126; in Lacroix, *Calc. Diff.* tom. I. page 264; or more particularly iu the latter part of Woodhouse's Trigonometry, he must allow the method here given to be at least extremely simple. The method of Cagnoli is not difficult, but as it gives only one-half of each compound series, the result is incomplete, and ∴ false. The same objection holds against Woodhouse's method, by induction in his third chapter. For instance, from the series of Cagnoli or Woodhouse, we have

$$\cos x = \cos x - \frac{1}{4.\cos x} - \frac{1}{16.\cos^3 x} - \frac{1}{32.\cos^5 x}, \; \&c.$$

a false result, as all others would also be for every value of n.

From series (1) the result obtained is

$$\left.\begin{aligned}\cos x = \cos x &- \frac{1}{4.\cos x} - \frac{1}{16.\cos^3 x} - \&c.\\ &+ \frac{1}{4.\cos x} + \frac{1}{16.\cos^3 x} + \&c.\end{aligned}\right\}$$

This imperfection deserves to be attended to, for it clearly shews how little dependance we ought to have on an induction gathered from particular values. In fact, as long as we only consider particular values, they may introduce into the calculus, reductions or simplifications which have no place in general.

The nature of the compound series (1) is such, that *for* n *an integer*, the indices continually diminish, and some of the terms at length vanish, but they again return in the first member of the

equation, and are after that destroyed by the terms of the second series, so that in this case the second series is useful, merely as shewing that the first is finite.

This will appear obvious, by writing the series (1) in the following form:

$$2.\cos nx=(2p)^n-\frac{n}{1}.(2p)^{n-2}-\frac{(3-n).n}{1.2}.(2p)^{n-4}-\frac{(5-n)(4-n).n}{1.2.3}.(2p)^{n-6}$$

$$\ldots\ldots\ldots\ldots\ldots-\frac{(2m-1-n)(2m-2-n)\ldots(m+1-n).n}{1.2.3\ldots\ldots\ldots m}.(2p)^{n-2m}$$

From this state of the series, we may perceive that the negative exponents will not begin to appear until $2m>n$; but there will always occur in the numerator of the general term an evanescent factor, as long as m will be comprised between n and $\frac{1}{2}n$; make then

$$m=n,\quad m=n+1,\quad m=n+2, \text{ \&c.}$$

and we obtain in order the terms

$$-(2p)^{-n},\ n(2p)^{-n-2},\ -\frac{n.(n+3)}{2}.(2p)^{-n-4}, \text{ \&c.}$$

which are the same as the terms of the second series with their signs changed.

This agreement of terms does not happen when n is fractional, and then all the terms of each series enter at the same time into the expression for $\cos nx$, which then goes on to infinity.

Euler first explained the difficulty presented by the developement of $\cos nx$. He proceeded by the integral calculus; and Lagrange subsequently, by the theory of derivative functions, advanced the subject much farther.

28. We shall now proceed to develope sines and cosines of multiple arcs in ascending series. As before, we have

$$2\cos nx = (p+\sqrt{p^2-1})^n + (p-\sqrt{p^2-1})^n$$

The former of these expressions may be written

$$(\sqrt{-1})^n . \left\{\sqrt{1-p^2} + \frac{p}{\sqrt{-1}}\right\}^n$$

and the latter,

$$(-\sqrt{-1})^n . \left\{\sqrt{1-p^2} - \frac{p}{\sqrt{-1}}\right\}^n .$$

Let us proceed to develope these expressions separately. By the binomial Theorem we have

$$(\sqrt{-1})^n . \left\{\sqrt{1-p^2} + \frac{p}{\sqrt{-1}}\right\}^n = (\sqrt{-1})^n \left\{(\sqrt{1-p^2})^n + \right.$$

$$n . (\sqrt{1-p^2})^{n-1} . \frac{p}{\sqrt{-1}} + n . \frac{n-1}{2} . (\sqrt{1-p^2})^{n-2} . \frac{p^2}{(\sqrt{-1})^2} +$$

$$\left.\frac{n . n-1 . n-2}{1 . 2 . 3} (\sqrt{1-p^2})^{n-3} . \frac{p^3}{(\sqrt{-1})^3} + \&c.\right\}$$

Denoting the several consecutive terms of the series within the brackets, in the right hand member of this equation, by the characters (A), (B), (C), &c. we have, by the binomial Theorem, the following series:

$$(A) = 1 - \frac{n}{2} . p^2 + \frac{n}{2} . \frac{n-2}{4} . p^4 - \frac{n}{2} . \frac{n-2}{4} . \frac{n-4}{6} . p^6 + \&c.$$

$$(B) = \frac{np}{\sqrt{-1}} . \left\{1 - \frac{n-1}{2} . p^2 + \frac{n-1}{2} . \frac{n-3}{4} . p^4 - \frac{n-1}{2} . \frac{n-3}{3} . \frac{n-5}{6} . p^6\right\}$$

$$(C) = n . \frac{n-1}{2} . \frac{p^2}{(\sqrt{-1})^2} . \left\{1 - \frac{n-2}{2} . p^2 + \frac{n-2}{2} . \frac{n-4}{4} . p^4 - \&c.\right\}$$

$$(D) = \frac{n . n-1 . n-2}{1 . 2 . 3} . \frac{p^3}{(\sqrt{-1})^3} . \left\{1 - \frac{n-3}{2} . p^2 + \frac{n-3}{2} . \frac{n-5}{4} . p^4 - \right\}$$

$$(E)=\frac{n.n-1.n-2.n-3}{1.2.3.4}\cdot\frac{p^4}{(\sqrt{-1})^4}\left\{1-\frac{n-4}{2}\cdot p^2+\frac{n-4.n-6}{2\quad\ 4}\cdot p^4\right\}$$

$$(F)=\frac{n.n-1.n-2.n-3.n-4}{1.2.3.4.5}\cdot\frac{p^5}{(\sqrt{-1})^5}\cdot\left\{1-\frac{(n-5)}{2}\cdot p^2+\&c.\right\}$$

$$(G)=\frac{n.(n-1).(n-2).(n-3).(n-4).(n-5)}{1.2.3.4.5.6}\cdot\frac{p^6}{(\sqrt{-1})^6}\cdot\left\{1-\frac{(n-6)}{2}p^2\right\}$$

$$= \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Collecting from all these series the like powers of p, and arranging accordingly, we find

$$(p+\sqrt{p^2-1})^n=(\sqrt{-1})^n.\ \left\{1+\frac{np}{\sqrt{-1}}-\frac{n^2}{2}\cdot p^2-\right.$$
$$\frac{n.(n^2-1)}{2.3}\cdot\frac{p^3}{\sqrt{-1}}+\frac{n^2(n^2-4)}{2.3.4}p^4+\frac{n.(n^2-1).(n^2-9)}{2.3.4.5}\cdot\frac{p^5}{\sqrt{-1}}-$$
$$\left.\frac{n^2.(n^2-4).(n^2-16)}{2.3.4.5.6}\cdot p^6+\&c.\right\}$$

Or dividing this expression into two series, we find

$$(p+\sqrt{p^2-1})^n=(\sqrt{-1})^n.\left\{1-\frac{n^2}{2}\cdot p^2+\frac{n^2.(n^2-4)}{2.3.4}\cdot p^4-\&c.\right\}$$
$$+n.(\sqrt{-1})^{n-1}.\left\{p-\frac{(n^2-1)}{2.3}\cdot p^3+\frac{(n^2-1).(n^2-9)}{2.8.4.5}\cdot p^5-\&c.\right\}$$

Since

$$(p-\sqrt{p^2-1})^n=(p+\sqrt{p^2-1})^{-n}$$

we need but change the sign of n in the preceding developement. This change will not affect the parts within the brackets, but the imaginary co-efficients will be changed in form. Changing the sign of n in

$$(\sqrt{-1})^n,$$

we get

$$(\sqrt{-1})^{-n};$$

but
$$(\sqrt{-1})^{-n} = (-\sqrt{-1})^n.$$

Again,
$$-n(\sqrt{-1})^{-n-1} = n.(-\sqrt{-1})^{n-1};$$

∴ we have

$$(p-\sqrt{p^2-1})^n = (-\sqrt{-1})^n.\left\{1-\frac{n^2}{2}.p^2+\frac{n^2(n^2-4)}{2.\,3.\,4}.p^4\ \&c.\right\}$$
$$+n.(-\sqrt{-1})^{n-1}.\left\{p-\frac{(n^2-1)}{2.\,3}.p^3+\frac{(n^2-1).(n^2-9)}{2.\,3.\,4.\,5}.p^5-\&c.\right\}$$

Adding together these two developements, we obtain

$$2\cos nx = \left\{(\sqrt{-1})^n+(-\sqrt{-1})^n\right\}.\left\{1-\frac{n^2}{2}.p^2+\right.$$
$$\left.\frac{n^2(n^2-4)}{2.\,3.\,4}.p^4-\&c.\right\}+n.\left\{(\sqrt{-1})^{n-1}+(-\sqrt{-1})^{n-1}\right\}.$$
$$\left\{p-\frac{(n^2-1)}{2.\,3}.p^3+\frac{(n^2-1).(n^2-9)}{2.\,3.\,4.\,5}.p^5-\&c.\right\}$$

The imaginary co-efficients can be very elegantly transformed by the Theorem of De Moivre. By this Theorem we have

$$(\cos A\pm\sqrt{-1}.\sin A)^n = \cos nA\pm\sqrt{-1}.\sin nA;$$

making $A=\frac{\pi}{2}$, we have

$$(\pm\sqrt{-1})^n = \cos\frac{n.\pi}{2}\pm\sqrt{-1}.\sin\frac{n\pi}{2},\ \therefore(\sqrt{-1})^n+$$
$$(-\sqrt{-1})^n = 2.\cos\frac{n\pi}{2}.$$

Similarly we have

$$(\sqrt{-1})^{n-1}+(-\sqrt{-1})^{n-1} = 2.\cos\frac{(n-1).\pi}{2}$$

∴ we have

$$\left.\begin{array}{l}\cos nx = \cos\frac{n\pi}{2}\cdot\left\{1-\frac{n^2}{2}\cdot p^2+\frac{n^2.(n^2-4)}{2.3.4}\cdot p^4-\&c\right\}\\ n.\cos\frac{(n-1)\pi}{2}\left\{p-\frac{(n^2-1)}{2.3}\cdot p^3+\frac{(n^2-1)(n^2-9)}{2.3.4.5}\cdot p^5-\&c.\right\}\end{array}\right\}(A)$$

Thus the question has been completely discussed by pure Algebra, without the aid of any higher calculus, and by a method too, that, for brevity and clearness, will not at least suffer by comparison with any method heretofore applied to the discussion of this difficult object.

Both members of the developement (A) are required to express the true value of $\cos nx$ when n is not an integer.

If n be an integer, and an even number,

$$\cos nx = \pm\left\{1-\frac{n^2}{2}\cdot p^2+\frac{n^2.(n^2-4)}{1.2.3}\cdot p^4-\&c.\right\} \qquad (1)$$

If odd,

$$\cos nx = \pm n.\left\{p-\frac{(n^2-1)}{2.3}\cdot p^3+\frac{(n^2-1).(n^2-9)}{2.3.4.5.6}\cdot p^5-\&c.\right\}(2)$$

The upper or lower sign is to be used in the former, as n is *pariter par*, or *impariter par*; in the latter, as $(n-1)$ is *pariter par*, or *impariter par.*

Differentiating the two members of equation, (A), we find

$$\left.\begin{array}{l}\sin nx = q.\cos\frac{n\pi}{2}\cdot\left\{-np+\frac{n.(n^2-4)}{2.3}\cdot p^3-\&c.\right\}\\ +q.\cos\frac{(n-1)\pi}{2}\cdot\left\{1-\frac{(n^2-1)}{2}\cdot p^2+\frac{(n^2-1).(n^2-9)}{2.3.4}\cdot p^4-\right\}\end{array}\right\}(B)$$

In this, as in the former, if n be not an integer, all the terms will be required to express the complete developement. If n be an integer and an even number,

$$\sin nx = \pm q. \left\{ -np + \frac{n.(n^2-4)}{2.\,3}.\,p^4 - \&c. \right\} \qquad (3)$$

If odd,

$$\sin nx = \pm q. \left\{ 1 - \frac{(n^2-1)}{2}.\,p^2 + \frac{(n^2-1).(n^2-9)}{2.\,3.\,4}.\,p^4 - \&c. \right\} \quad (4)$$

These four series may easily be converted into others, as follows. Series (1), if we substitute $(\frac{1}{2}\pi - x)$ for x, and recollect that n is an even number, is converted into the following:

$$\cos nx = \pm \left\{ 1 - \frac{n^2}{2}.\,q^2 + \frac{n^2\,(n^2-4)}{2.\,3.\,4}.\,q^4 - \&c. \right\} \qquad (5)$$

Series (2) similarly treated, since n is an odd number, gives

$$\sin nx = \pm n \left\{ q - \frac{(n^2-1)}{2.\,3}.\,q^3 + \frac{(n^2-1).(n^2-9)}{2.\,3.\,4.\,5.\,6}\,q^5 - \&c. \right\} \qquad (6)$$

From series (3) and (4) we have

$$\sin nx = \pm p. \left\{ -nq + \frac{n.\,(n^2-4)}{2.\,3}.\,q^3 - \frac{n.\,(n^2-4).\,(n^2-16)}{2.\,3.\,4.\,5}\,q^5 + \right\} \quad (7)$$

$$\cos nx = \pm p. \left\{ 1 - \frac{(n^2-1)}{2}.\,q^2 + \frac{(n^2-1).\,(n^2-9)}{2.\,3.\,4}.\,q^4 - \&c. \right\} \quad (8)$$

Thus we find in all, fourteen series for expressing sines and cosines of multiple arcs, independant of the two that have been given in (Art. 3.) of this chapter, thus furnishing as complete and as uniform a collection as any that has yet appeared on the same subject.

The enumeration here given, accounts completely for the different results obtained for sines and cosines of multiple arcs by different authors. Previously to the subject having undergone a complete investigation, the most comprehensive series was that given by Newton.* It was as follows:—

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* *Epistolâ ad Oldemburgium, datâ anno* 1676. Of this series

Let d be the diameter of a circle, a the subtense of the arc $(2z)$, and m the subtense of the multiple arc $(2nz)$, then

$$m=na+\frac{1-n^2}{2.\,3}\cdot\frac{a^2}{d^2}\cdot A+\frac{(9-n^2)}{4.\,5}\cdot\frac{a^2}{d^2}\cdot B+\&c.$$

where A, B, &c. denote the first, second, &c. terms of this series respectively. This series was resolved into two by Euler, and also implicitly by Simpson, as is shewn by Frisius in his Cosmographia.

29. We shall now proceed to the second of the problems announced in (Art. 27.) of this chapter.

It is essential in the integral calculus to be furnished with expansions of the n^{th} powers of sines and cosines, and these expansions it is the proper business of Trigonometry to furnish. We have already discovered that

$$2\cos A=e^{A\sqrt{-1}}+e^{-A\sqrt{-1}},$$

and

$$2.\sin A=\frac{1}{\sqrt{-1}}\left(e^{A\sqrt{-1}}-e^{-A\sqrt{-1}}\right).$$

Denoting $e^{A\sqrt{-1}}$ by the character x, for the sake of brevity, we have

$$2^n\cos^n A=\left(x+\frac{1}{x}\right)^n,$$

and

$$2^n\sin^n A=\frac{1}{(\sqrt{-1})^n}\cdot\left(x-\frac{1}{x}\right)^n.$$

Expanding,

$$2^n.\cos^n A=x^n+n.\,x^{n-2}+n.\,\frac{n-1}{2}\cdot x^{n-4}\ldots\ldots\ldots+\frac{n}{x^{n-2}}+\frac{1}{x^n}$$

Paulus Frisius makes the following remark:—“ Elegantissimam et fundamentalem hanc formulam quæ Bernouillianas omnes, Simpsonianas, et Eulerianas simul complectitur, primus omnium exhibuit Newtonus.”—*Pauli Frisii Cosmographia*, tom. I. page 171.

or, collecting into pairs the terms beginning at each extremity, we have, if n be even,

$$2^n.\cos^n A = \left(x^n + \frac{1}{x^n}\right) + n.\left(x^{n-2} + \frac{1}{x^{n-2}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{n.\,n-1.\,n-2\ldots\ldots\left(n-\frac{n}{2}+2\right)}{1.\,2.\,3.\,\ldots\ldots\,\left(\frac{n}{2}-1\right)}\left(x+\frac{1}{x}\right)+\frac{n.\,n-1.\,n-2\ldots\ldots\left(n-\frac{n}{2}+1\right)}{1.\,2.\,3\,\ldots\ldots\ldots\,\frac{n}{2}}$$

But from the nature of x,

$$2\cos nA = x^n + \frac{1}{x^n}$$

$\therefore$ this becomes

$$2^n.\cos^n A = 2.\left\{\cos nA + n.\cos(n-2)A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right.$$

$$\left.\frac{n.\,n-1.\,n-2\ldots\ldots\ldots\left(\frac{n}{2}+2\right)}{1.\,2.\,3\ldots\ldots\ldots\ldots\,\frac{n}{2}-1}\cos A\right\} + \frac{n.\,n-1.\,n-2\ldots\ldots\ldots\ldots\left(\frac{n}{2}+1\right)}{1.\,2.\,3\ldots\ldots\ldots\,\frac{n}{2}}.$$

This last term may be written

$$2^{\frac{n}{2}}\,\frac{n.\,n-1.\,n-2\ldots\ldots\ldots\left(\frac{n}{2}+1\right)}{2.\,4.\,6\ldots\ldots\ldots\ldots\,n}.\,\frac{1.\,3.\,5.\,\&c.\ldots\ldots.(n-1)}{1.\,3.\,5.\ldots\ldots\ldots\ldots\,(n-1)},$$

which, by erasing the common factors in the numerator and denominator, and incorporating the odd digits in their natural places, becomes

$$2^{\frac{n}{2}}.\,\frac{1.\,3.\,5\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n-1)}{1.\,2.\,3\ldots\ldots\ldots\ldots\ldots\ldots\,\frac{n}{2}}.$$

Hence the reader may deduce that

$$2^{\frac{n}{2}} = \frac{n.\,n-1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\left(\frac{n}{2}+1\right)}{1.\,3.\,5\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,(n-1)}.$$

If n be odd, $2^n . \cos^n A =$

$$\left(x^n + \frac{1}{x^n}\right) + n.\left(x^{n-2} + \frac{1}{x^{n-2}}\right) \ldots \frac{n.\, n-1.\, n-2 \ldots \left(\frac{n+5}{2}\right)}{1.\, 2.\, 3 \ldots\ldots \frac{n-3}{2}} \left(x^2 + \frac{1}{x^2}\right) +$$

$$\frac{n.\, n-1.\, n-2 \ldots\ldots\ldots\ldots \left(\frac{n+3}{2}\right)}{1.\, 2.\, 3 \ldots\ldots\ldots\ldots\ldots\ldots \frac{n-1}{2}} . \left(x + \frac{1}{x}\right)$$

$\therefore$, as before, we have $2^n . \cos^n A =$

$$2. \left\{\cos n A + n. \cos (n-2) A + \ldots \frac{n.\, n-1.\, n-2 \ldots\ldots \frac{n+3}{2}}{1.\, 2.\, 3 \ldots\ldots\ldots \frac{n-1}{2}} . \cos A\right\}$$

Again, $2^n . \sin^n A$ can be likewise expanded, but differently, as n is an odd or an even number.

If n be even, we have

$$2^n . \sin^n A = \frac{1}{(\sqrt{-1})^n} \left\{x^n - n.\, x^{n-2} + n. \frac{n-1}{2} . x^{n-4} \ldots\ldots - \frac{n}{x^{n-2}} + \frac{1}{x^n}\right\}$$

$$= \frac{1}{(\sqrt{-1})^n} \left\{\left(x^n + \frac{1}{x^n}\right) - n.\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \ldots \frac{n \ldots\ldots \left(\frac{n}{2}+2\right)}{1 \ldots\ldots \left(\frac{n}{2}-1\right)} \left(x + \frac{1}{x}\right)\right\}$$

$$\pm \frac{1}{(\sqrt{-1})^n} \frac{n.\, n-1.\, n-2 \ldots\ldots\ldots\ldots \left(\frac{n}{2}+1\right)}{1.\, 2.\, 3 \ldots\ldots\ldots\ldots\ldots\ldots \frac{n}{2}} .$$

or writing $\cos n A$ for $x^n + \frac{1}{x^n}$, we have

$$2^n \sin^n A = \pm 2\left\{\cos nA - n.\cos(n-2)A + \ldots \pm \frac{n..\left(\frac{n}{2}+2\right)}{1..\left(\frac{n}{2}-1\right)}.\cos A\right\}$$

$$+\frac{n.\,n-1.\,n-2.\ldots\ldots\ldots\left(\frac{n}{2}+1\right)}{1.\,2.\,3.\ldots\ldots\ldots\ldots\ldots\ldots\frac{n}{2}}.$$

The upper sign being used if n be *pariter par*, and the lower if *impariter par.*

The last term is obviously positive in both cases, since in the former its place is *odd*, and ∴ positive, and it continues so when multiplied by $\frac{1}{(\sqrt{-1})^n}$; and in the latter its place is *even*, and ∴ *negative*, and it becomes positive when multiplied by $\frac{1}{(\sqrt{-1})^n}$. This is obvious, since the *pariter par* powers of this quantity are positive, and the *impariter par* negative

If n be odd, we have $2^n.\sin^n A =$

$$\frac{1}{(\sqrt{-1})^n}\left\{\left(x^n - \frac{1}{x^n}\right) - n.\left(x^{n-2} - \frac{1}{x^{n-2}}\right)\ldots \pm \frac{n\ldots\ldots\frac{n+3}{2}}{1\ldots\ldots\frac{n-1}{2}}\left(x - \frac{1}{x}\right)\right\}$$

or, $2^n.\sin^n A =$

$$\frac{1}{(\sqrt{-1})^{n-1}}\left\{\frac{1}{\sqrt{-1}}\left(x^n - \frac{1}{x^n}\right)\ldots \pm \frac{1}{\sqrt{-1}}.\frac{n.\ldots\ldots\frac{n+3}{2}}{1\ldots\ldots\frac{n-1}{2}}\left(x - \frac{1}{x}\right)\right\}$$

But from the nature of x, it has been already shewn that

$$\sin nA = \frac{1}{\sqrt{-1}}\left(x^n - \frac{1}{x^n}\right)$$

∴ we have $2^n.\sin^n A =$

$$\pm 2 \left\{ \sin nA - n. \sin(n-2)A + \&c. \ldots\ldots\ldots \pm \frac{n \ldots\ldots \frac{n+3}{2}}{1 \ldots\ldots \frac{n-1}{2}}. \sin A \right\}$$

according as $(n-1)$ is *pariter* or *impariter par*, or according as n is one of the series 1, 5, 9, 13, 17, &c. or of the series 3, 7, 11, 15, &c. we use the upper or the lower sign.

30. The preceding method only applies to the case when n is an integer, and positive. It would be easy to adapt it to any case, by attending to the fact that $\left(\frac{1}{x} + x\right)^n = \left(x + \frac{1}{x}\right)^n$, and by taking the two developements and adding them, &c. We shall, however, for the general case, prefer the following simple and elegant method, which is altogether due to Lagrange.

Let $y = \cos^n x$, then differentiating, we have $dy = -n. \cos^{n-1} x. \sin x. dx$, whence we have

$$ny. \sin x + \frac{dy}{dx}. \cos x = 0.$$

Let us then assume

$$y = A. \cos mx + B. \cos(m-1)x + C. \cos(m-2)x + D. \cos(m-3)x + \&c.$$

where A, B, C, &c. designate indeterminate co-efficients. Differentiating this, and then substituting for y and dy in the above differential equation, it becomes

$$\left.\begin{array}{l} n. \sin x. \{A. \cos mx + B. \cos(m-1)x + C. \cos(m-2)x + \&c.\} \\ -\cos x \{mA. \sin mx + (m-1)B. \sin(m-1)x + (m-2). \\ \quad C. \sin(m-2). x\} \end{array}\right\} = 0$$

If we observe that

$$\sin x. \cos mx = \tfrac{1}{2}. \{\sin(m+1)x - \sin(m-1)x\},$$

and

$$\cos x. \sin mx = \tfrac{1}{2}. \{\sin(m+1)x + \sin(m-1). x\};$$

and that we apply this transformation to all such products, we obtain, arranging relatively to the sines of the multiples of x,

$$\left.\begin{array}{l}\{nA-mA\}.\sin(m+1)x \\ +\{nB-(m-1)B\}.\sin mx \\ +\{nC-nA-(m-2)C-mA\}.\sin(m-1)x \\ +\{nD-nB-(m-3)D-(m-1)B\}.\sin(m-2)x \\ +\&c.\end{array}\right\}=0$$

Equating to zero the co-efficient of each sine separately, we obtain,

$$(n-m).A=0$$

$$(n-m+1).B=0$$

$$(n-m+2).C-(n+m).A=0$$

$$(n-m+3).D-(n+m-1).B=0$$

$$(n-m+4).E-(n+m-2).C=0$$

The two first of these equations cannot be satisfied by making $A=0$ and $B=0$, for this supposition would render nothing all the subsequent co-efficients; but by making $m=n$ the first equation may be verified, A remaining indeterminate, and the second gives $B=0$; thence we have

$$C=\frac{2n}{2}.A;\ D=\frac{2n-1}{3}.B;\ E=\frac{2n-2}{4}.C;\ \&c.$$

Introducing all these values in the series from which we set out, we obtain

$$\cos^n x=A.\left\{\cos nx+\frac{n}{1}.\cos(n-2)x+\frac{n.(n-1)}{1.2}.\cos(n-4)x+\&c.\right\}$$

to determine A, let $x=0$, then we have

$$1 = A. \left\{1+\frac{n}{1}+\frac{n.(n-1)}{1.2}+\&c.\right\} = A.(1+1)^n = A.2^n$$

$$\therefore A = \frac{1}{2^n}$$

When n is a positive integer, the values beginning at each end unite in pairs, so that each side of the equation is then divisible by 2, and thus we obtain the series already given.

The series for the sine may be deduced *a priori* like that for the cosine; or may be had from the latter, by using $(\frac{1}{2}\pi - x)$ for x.

31. It may be necessary to remark, before quitting this chapter, that if the imaginary formulæ given in Art. (5) had been proved by the differential calculus, the whole of Trigonometry might have been founded thereon, independantly of Geometrical considerations. Multiply the expressions for $\sin x$ and $\cos z$; multiply also these for $\cos x$ and $\sin z$; add the products, and the result is

$$\sin x.\cos z + \cos x.\sin z = \frac{e^{(x+z)\sqrt{-1}} - e^{-(x+z)\sqrt{-1}}}{2\sqrt{-1}} = \sin(x+z);$$

and from the value of $\sin(x+z)$ thus obtained, may be deduced analytically all the formulæ of Trigonometry. This reasoning is, however, sophistical and illusory at bottom, as there is no known method of differentiating a sine or cosine independantly of geometrical considerations, unless we assume the developement of $\sin(x+z)$.

CHAP. III.

DECOMPOSITION OF TRIGONOMETRICAL LINES INTO FACTORS, IN FUNCTIONS OF THEIR RESPECTIVE ARCS—CONSEQUENCES DEDUCIBLE THEREFROM—APPLICATION OF THE TRIGONOMETRICAL CALCULUS TO THE DECOMPOSITION INTO FACTORS OF CERTAIN ALGEBRAIC FUNCTIONS—THEOREMS OF 'COTES AND DE MOIVRE.'

1. Formulæ for the sine and cosine in terms of the arc have been already obtained in the preceding chapter, we shall accordingly proceed to determine the roots of the equations,

$$0 = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.,$$

and

$$0 = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

or in other words, to decompose $\sin x$ and $\cos x$ into their factors. Now since $\sin x$ vanishes on the several suppositions of $x=0$; $x=\pm\pi$; $x=\pm 2\pi$, &c. it follows that x; $1+\frac{x}{\pi}$; $1-\frac{x}{\pi}$; $1+\frac{x}{2\pi}$; $1-\frac{x}{2\pi}$; &c. must be factors of $\sin x$. Again, since these are the only values that render $\sin x$ zero, its expression can admit no other factors, functions of x, and therefore we must have

$$\sin x = A.x.\left(1-\frac{x^2}{\pi^2}\right).\left(1-\frac{x^2}{4\pi^2}\right) \&c.\ \textit{ad inf.}$$

Now as x diminishes, $\frac{\sin x}{x}$ approaches unity as its limit, and therefore we must have $A=1$, and hence

$$\sin x = x . \left(1 - \frac{x^2}{\pi^2}\right) . \left(1 - \frac{x^2}{4\pi^2}\right), \text{ \&c.}$$

Similarly since $\cos x$ vanishes on the several suppositions

$$x = \pm \frac{\pi}{2};\ x = \pm \frac{3\pi}{2}, \text{ \&c.}$$

and approaches unity as its limit when x decreases to zero, we must have

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right) . \left(1 - \frac{4x^2}{9\pi^2}\right) . \left(1 - \frac{4x^2}{25\pi^2}\right) \text{ \&c.}$$

2. These singular results are susceptible of very many transformations, and admit of several important consequences, which we shall now proceed to discuss.

From the value already obtained for $\sin x$, we have

$$\sin \frac{m\pi}{n} = \frac{m\pi}{n}\left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{4n^2}\right)\left(1 - \frac{m^2}{9n^2}\right)\left(1 - \frac{m^2}{16n^2}\right), \text{ \&c.}$$

If in this formula we change $\frac{m\pi}{n}$ into $\frac{\pi}{2} - \frac{m\pi}{n}$, we shall have

$$\sin\left(\frac{\pi}{2} - \frac{m\pi}{n}\right) = \cos \frac{m\pi}{n} =$$

$$\pi\left(\tfrac{1}{2} - \frac{m}{n}\right) \frac{\left(\frac{1}{2} + \frac{m}{n}\right)\left(\frac{3}{2} - \frac{m}{n}\right)\left(\frac{3}{2} + \frac{m}{n}\right)\left(\frac{5}{2} - \frac{m}{n}\right)\left(\frac{5}{2} + \frac{m}{n}\right)}{1.1 \qquad 2.2 \qquad 3.3}, \text{ \&c.} =$$

$$\frac{\pi}{2} \frac{\left(1 - \frac{2m}{n}\right)\left(1 + \frac{2m}{n}\right)\left(3 - \frac{2m}{n}\right)\left(3 + \frac{2m}{n}\right)\left(5 - \frac{2m}{n}\right)\left(5 + \frac{2m}{n}\right)}{2.2 \qquad 4.4 \qquad 6.6}, \text{ \&c.}$$

$$= \frac{\pi}{2} . \frac{1.3.3.5.5.7.7.\text{\&c.}}{2.2.4.4.6.6.\text{\&c.}} \left(1 - \frac{4m^2}{n^2}\right)\left(1 - \frac{4m^2}{9n^2}\right)\left(1 - \frac{4m^2}{25n^2}\right), \text{ \&c.}$$

But from the expression already found for $\cos x$, $\left(\text{using for } x, \frac{m\pi}{n}\right)$ we have

$$\cos\frac{m\pi}{n}=\left(1-\frac{4m^2}{n^2}\right)\left(1-\frac{4m^2}{9n^2}\right)\left(1-\frac{4m^2}{25n^2}\right), \text{ \&c.}$$

Comparing these values for $\cos\frac{m\pi}{n}$, we have

$$\frac{\pi}{2}\cdot\frac{3.\,3.\,5.\,5.\,7.\,7.\text{ \&c.}}{2.\,2.\,4.\,4.\,6.\,6.\text{ \&c.}}=1,\ \therefore\frac{\pi}{2}=\frac{2.\,2.\,4.\,4.\,6.\,6.\text{ \&c.}}{3.\,3.\,5.\,5.\,7.\,7.\text{ \&c.}}$$

which singular value is due to Wallis.—(*Vide opera ejus*, page 256, Vol. II.)

These values for the sine and cosine are due to the celebrated Euler, who deduces from them very many important consequences. The logarithmic values of the sine and cosine can immediately and rapidly be formed from these series, as the reader will find in an after part of this treatise.

3. Using for $\sin x$ and $\cos x$ their exponential values, we have

$$\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}=x.\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{4\pi^2}\right)\cdot\left(1-\frac{x^2}{9\pi^2}\right)\text{\&c.}$$

or making $x=\frac{u}{\sqrt{-1}}$, we have

$$\frac{e^u-e^{-u}}{2}=u.\left(1+\frac{u^2}{\pi^2}\right)\cdot\left(1+\frac{u^2}{4\pi^2}\right).\left(1+\frac{u^2}{9\pi^2}\right)\text{\&c.}=\frac{u}{1}+\frac{u^3}{1.2.3.}+\frac{u^5}{1.\,2.\,3.\,4.\,5}+\text{ \&c.}$$

Again, from the value of the cosine, we obtain

$$\frac{e^u+e^{-u}}{2}=\left(1+\frac{4u^2}{\pi^2}\right)\left(1+\frac{4u^2}{9\pi^2}\right)\left(1+\frac{4u^2}{16\pi^2}\right)\text{\&c.}=1+\frac{u^2}{1.2}+\frac{u^4}{1.2.3.4}+\text{\&c.}$$

These developements lead very simply to the summation of some numerical series. In general, if any series $1+Az+Bz^2+Cz^3+$&c.$=(1+\alpha z)\,(1+\beta z)\,(1+\gamma z)$ &c., then from the nature of equations,

$$A=\alpha+\beta+\gamma+\delta+\text{\&c.}$$

$$B=\alpha\beta+\alpha\gamma+\beta\gamma+\text{\&c.}$$

$$C=\alpha\beta\gamma+\alpha\gamma\delta+\beta\gamma\delta+\text{\&c.}$$

&c. &c. &c.

From these, by the nature of equations also, we are furnished with expressions for the sums of the powers of α, β, γ, &c. as follows:—

$S_1=A$, $S_2=A^2-2B$, $S_3=A^3-3AB+3C$, $S_4=A^4-4A^2B+4AC-5D+2B^2$, &c.

Returning now to the expansions, and using for u^2, $\pi^2 z$, we find

$$1+\frac{\pi^2 z}{1.\,2.\,3}+\frac{\pi^4 z^2}{1.\,2.\,3.\,4.\,5}+\text{\&c.}=(1+z)(1+\tfrac{1}{4}z)(1+\tfrac{1}{9}z)(1+\tfrac{1}{16}z)\text{\&c.};$$

comparing this with the theory just given, we have

$$A=\frac{\pi^2}{1.\,2.\,3},\ B=\frac{\pi^4}{1.\,2.\,3.\,4.\,5},\ \text{\&c.}$$

and

$$\alpha=1,\ \beta=\tfrac{1}{4},\ \gamma=\tfrac{1}{9},\ \text{\&c.}$$

∴ we have the following series:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{\&c.}=\frac{\pi^2}{6}$$

$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\text{\&c.}=\frac{\pi^4}{90}$$

$$1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\text{\&c.}=\frac{\pi^6}{945}$$

$$1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\&c. = \frac{\pi^8}{9450}$$

$$1+\frac{1}{4^5}+\frac{1}{9^5}+\frac{1}{16^5}+\&c. = \frac{\pi^{10}}{93555}$$

&c. &c. &c.

This summation may be continued to an indefinite extent, and from it we may perceive that the sum of the series $1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\frac{1}{4^{2m}}+\&c.$ depends on the perimeter of the circle.

The series for $\frac{e^u+e^{-u}}{2}$ using for u^2, $\pi^2 z$, gives

$$1+\frac{\pi^2 z}{1.2.4}+\frac{\pi^4 z^2}{1.2.3.4.4^2}+\frac{\pi^6 z^3}{1.2.3.4.5.6.4^3}+\&c. = (1+z)(1+\tfrac{1}{9}z)(1+\tfrac{1}{25}z)(1+\tfrac{1}{49}z)\ \&c.$$

Comparing this with the general theory, we have

$$A=\frac{\pi^2}{1.2.4},\ B=\frac{\pi^4}{1.2.3.4.4^2},\ C=\frac{\pi^6}{1.2.3.4.5.6.4^3}$$

Hence we have

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\ \&c. = \frac{1}{1}.\frac{\pi^2}{2^3}$$

$$1+\frac{1}{9^2}+\frac{1}{25^2}+\frac{1}{49^2}\ \&c. = \frac{2}{1.2.3}.\frac{\pi^4}{2^5}$$

$$1+\frac{1}{9^3}+\frac{1}{25^3}+\frac{1}{49^3}\ \&c. = \frac{16}{1.2.3.4.5}.\frac{\pi^6}{2^7}$$

$$1+\frac{1}{9^4}+\frac{1}{25^4}+\frac{1}{49^4}\ \&c. = \frac{272}{1.2.3.4.5.6.7}.\frac{\pi^8}{2^9}$$

&c. &c. &c.

Thus we can find the sum of $1+\frac{1}{3^{2m}}+\frac{1}{5^{2m}}+\frac{1}{7^{2m}}+$ &c. which depends as the preceding upon the perimeter of the circle. Taking

$$S=1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\frac{1}{4^{2m}}+\frac{1}{5^{2m}}+\text{\&c.}$$

and

$$S'=1+\frac{1}{3^{2m}}+\frac{1}{5^{2m}}+\frac{1}{7^{2m}}+\text{\&c.}$$

Denoting the sum of the even reciprocal powers by Σ, we find by subtraction

$$\Sigma=\frac{1}{2^{2m}}+\frac{1}{4^{2m}}+\frac{1}{6^{2m}}+\text{\&c.}=\frac{1}{2^{2m}}\left\{1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\text{\&c.}\right\}=\frac{S}{2^{m}},$$

Subtracting 2Σ, or $\frac{2S}{2^{m}}$ from S, we have

$$S\left(1-\frac{2}{2^{2m}}\right)=1-\frac{1}{2^{2m}}+\frac{1}{3^{2m}}-\frac{1}{4^{2m}}+\frac{1}{5^{2m}}-\text{\&c.}$$

with the signs alternately positive and negative.

4. From the decomposition of $\frac{e^{u}\pm e^{-u}}{2}$, we may deduce that of $\frac{e^{x}\pm e^{\pm y}}{2}$; for we have

$$1°.\ \frac{e^{x}+e^{y}}{2}=e^{\frac{1}{2}(x+y)}.\ \frac{e^{\frac{1}{2}(x-y)}+e^{-\frac{1}{2}(x-y)}}{2}=$$

$$e^{\frac{1}{2}(x+y)}.\left(1+\frac{(x-y)^{2}}{\pi^{2}}\right).\left(1+\frac{(x-y)^{2}}{9\pi^{2}}\right).\ \text{\&c.}$$

$$2°.\ \frac{e^{x}+e^{-y}}{2}=e^{\frac{1}{2}(x-y)}.\ \frac{e^{\frac{1}{2}(x+y)}+e^{-\frac{1}{2}(x+y)}}{2}=$$

$$e^{\frac{1}{2}(x-y)}.\left(1+\frac{(x+y)^{2}}{\pi^{2}}\right).\left(1+\frac{(x+y)^{2}}{9\pi^{2}}\right)\ \text{\&c.}$$

3°. $\frac{e^x - e^y}{2} = e^{\frac{1}{2}(x+y)} . \frac{e^{\frac{1}{2}(x-y)} - e^{-\frac{1}{2}(x-y)}}{2} =$

$$e^{\frac{1}{2}(x+y)} . \left(\frac{x-y}{2}\right) . \left(1+\frac{(x-y)^2}{4\pi^2}\right) . \left(1+\frac{(x-y)^2}{16\pi^2}\right) \&c.$$

4°. $\frac{e^x - e^{-y}}{2} = e^{\frac{1}{2}(x-y)} . \frac{e^{\frac{1}{2}(x+y)} - e^{-\frac{1}{2}(x+y)}}{2} =$

$$e^{\frac{1}{2}(x-y)} . \left(\frac{x+y}{2}\right) . \left(1+\frac{(x+y)^2}{4\pi^2}\right) . \left(1+\frac{(x+y)^2}{16\pi^2}\right) \&c.$$

If we make in these results $y=0$, we have

$$\frac{e^x + 1}{2} = e^{\frac{1}{2}x} . \left(1 + \frac{x^2}{\pi^2}\right) . \left(1 + \frac{x^2}{9\pi^2}\right) . \left(1 + \frac{x^2}{25\pi^2}\right) \&c.$$

$$\frac{e^x - 1}{2} = e^{\frac{1}{2}x} . \frac{x}{2} . \left(1 + \frac{x^2}{4\pi^2}\right) . \left(1 + \frac{x^2}{16\pi^2}\right) \&c.$$

From these values we have

$$\frac{e^{x+\frac{1}{2}y} + e^{-\frac{1}{2}y}}{e^x + 1} = \left(1 + \frac{2xy+y^2}{\pi^2 + x^2}\right) . \left(1 + \frac{2xy+y^2}{9\pi^2 + x^2}\right) \&c.$$

Let $x = u . \sqrt{-1}$, $\frac{1}{2}y = -v . \sqrt{-1}$, then we shall have

$$\frac{e^{x+\frac{1}{2}y} + e^{-\frac{1}{2}y}}{e^x + 1} = \frac{e^{(u-v)\sqrt{-1}} + e^{v\sqrt{-1}}}{e^{u\sqrt{-1}} + 1} = \frac{\cos v + \cos (u-v)}{1 + \cos u} =$$

$$\cos v + \tan \tfrac{1}{2}u . \sin v = \left(1 + \frac{4uv - v^2}{\pi^2 - u^2}\right) . \left(1 + \frac{4uv - v^2}{9\pi^2 - u^2}\right), \&c. =$$

$$\left(1 + \frac{2v}{\pi - u}\right) . \left(1 - \frac{2v}{\pi + u}\right) . \left(1 + \frac{2v}{3\pi - u}\right) . \left(1 - \frac{2v}{3\pi + u}\right), \&c.$$

Now if we substitute in the expression $\cos v + \tan \frac{1}{2}u . \sin v$, $\frac{m\pi}{n}$ for u, and $\frac{z\pi}{2n}$ for v, and reduce into series the quantities $\cos \frac{z\pi}{2n}$, and $\sin \frac{z\pi}{2n}$, we shall obtain

$$1+\frac{\pi z}{2n}.\tan\frac{m\pi}{2n}-\frac{\pi^2 z^2}{2.4.n^2}-\frac{\pi^3 z^3}{2.4.6.n^3}.\tan\frac{m\pi}{2n}+\&c.=$$

$$\left(1+\frac{z}{n-m}\right).\left(1-\frac{z}{n+m}\right).\left(1+\frac{z}{3n-m}\right).\left(1-\frac{z}{3n+m}\right)\&c.$$

From the identity of these two series we can, as in Art. 3. of this Chapter, find the sum, sum of squares, cubes, &c. of the series

$$\frac{1}{n-m}-\frac{1}{n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}+\&c.$$

and the sum is equal to $\frac{\pi}{2n}.\tan\frac{m\pi}{2n}$.

A similar investigation applied to $\frac{e^x-e^{-y}}{e^x-1}$ would give

$$\frac{\pi}{2n}.\cot\frac{m\pi}{2n}=\frac{1}{m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\&c.$$

Adding together these two sums, we have

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\&c.=\frac{\pi}{n}.\operatorname{cosec}\frac{m\pi}{n}.$$

Subtracting the same, we have

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}\&c.=\frac{\pi}{2}.\cot\frac{m\pi}{n}.$$

This subject might be carried on to a much greater extent, but such a proceeding would lead us too far from the direct object of the present treatise.

5. If in the expression already found for the sine, we write for x, $\frac{m\pi}{2n}$, we obtain

$$\sin\frac{m\pi}{2n}=\frac{m\pi}{2n}.\frac{2n-m}{2n}.\frac{2n+m}{2n}.\frac{4n-m}{4n}.\frac{4n+m}{4n},\&c.$$

hence we have

$$\frac{\pi}{2}=\sin\frac{m\pi}{2n}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4m}{4n+m}, \&c. \quad (A)$$

In which, let us suppose $m=n$, and we obtain

$$\frac{\pi}{2}=\frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ \&c.}{3.\ 3.\ 5.\ 5.\ 7.\ 7.\ \&c.}$$

which is the expression of Wallis, already otherwise obtained.

The expression for the cosine, using for x, $\frac{m\pi}{2n}$, becomes in like manner

$$\cos\frac{m\pi}{2n}=\frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}, \&c.$$

Dividing these expressions for the sine and cosine, and inverting them, we obtain, recollecting the value of $\frac{\pi}{2}$, the four following series:

$$\tan\frac{m\pi}{2n}=\frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}, \&c.$$

$$\cot\frac{m\pi}{2n}=\frac{n-m}{m}\cdot\frac{n+m}{2n-m}\cdot\frac{3n-m}{2n+m}\cdot\frac{3n+m}{4n-m}\cdot\frac{5n-m}{4n+m}, \&c.$$

$$\sec\frac{m\pi}{2n}=\frac{n}{n-m}\cdot\frac{n}{n+m}\cdot\frac{3n}{3n-m}\cdot\frac{3n}{3n+m}, \&c.$$

$$\operatorname{cosec}\frac{m\pi}{2n}=\frac{2n}{m\pi}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}, \&c.$$

or eliminating $\frac{\pi}{2}$ it becomes

$$\operatorname{cosec}\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{n}{2n-m}\cdot\frac{3n}{2n+m}\cdot\frac{3n}{4n-m}\cdot\frac{5n}{4n+m}, \&c.$$

These series may evidently be all written in two forms, reject-

ing or retaining the quantity $\frac{\pi}{2}$; for instance, by eliminating $\frac{\pi}{2}$, the above value for the sine may be written

$$\sin\frac{m\pi}{2n}=\frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\frac{4n+m}{4n}, \&c.$$

The reader will find them in both forms in pp. 450–452 of *Lacroix Traité de Calcul Diff. tom.* III.

If in series (A) we make $\frac{m}{n}=\frac{1}{2}$, we have $\sin\frac{1}{4}\pi=\frac{1}{\sqrt{2}}$, ∴ we have

$$\frac{\pi}{2}=\sqrt{2}.\frac{4.\ 4.\ 8.\ 8.\ 12.\ 12.\ \&c.}{3.\ 5.\ 7.\ 9.\ 11.\ \&c.}$$

from which, substituting, we obtain

$$\sqrt{2}=\frac{2.\ 2.\ 6.\ 6.\ 10.\ 10.\ \&c.}{1.\ 3.\ 5.\ 7.\ 9.\ 11.\ \&c.}$$

This singular expression has not been as yet otherwise obtained.

If we make $\frac{m}{n}=\frac{1}{3}$, we find

$$\frac{\pi}{2}=\frac{3}{2}\cdot\frac{6.\ 6.\ 12.\ 12.\ 18.\ 18.\ \&c.}{5.\ 7.\ 11.\ 13.\ 17.\ 19.\ 23.\ \&c.}$$

Other similar values and expressions might be obtained, *ad libitum*, but we shall pass to more interesting matters

6. Any one of the above formulæ might be so ordered as to give the Trigonometrical line for one arc if we know it for another; for instance, we have

$$\sin\frac{k\pi}{2n}=\frac{k}{n}\cdot\frac{2n-k}{n}\cdot\frac{2n+k}{n}, \&c.$$

$$\therefore \sin\frac{m\pi}{2n}=\sin\frac{k\pi}{2n}\cdot\frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot \&c.$$

7. The Trigonometrical calculus affords us means of decomposing into factors the Algebraic equations $y^{2n}-2y^n.\cos\theta+1=0$, and $y^n\pm1=0$. For this purpose we shall follow the very elegant and simple analysis of Lagrange. By what has been already proved, we may write

$$2\cos z=y+\frac{1}{y}, \text{ and } 2\cos nz=y^n+\frac{1}{y^n};$$

or,

$$y^2-2y\cos z+1=0, \text{ and } y^{2n}-2y^n\cos nz+1=0;$$

now from the mode of forming these equations, it is obvious that any value of y that satisfies the former of these equations, will also satisfy the latter; but the former of these has two roots, one the reciprocal of the other, which ∴ are also roots of the latter; ∴ the former equation is itself one of the divisors of the latter. Assuming $nz=2m\pi+\theta$, we have

$$\cos nz=\cos\theta, \text{ and } z=\frac{2m\pi+\theta}{n},$$

∴ the factors of the second degree of the equation, $y^{2n}-2y^n.\cos\theta+1=0$, are comprised in the formula

$$y^2-2y.\cos\frac{2m\pi+\theta}{n}+1=0,$$

∴ we have

$$\left.\begin{aligned}&y^{2n}-2y^n.\cos\theta+1=\left(y^2-2y.\cos\frac{\theta}{n}+1\right).\\&\left(y^2-2y.\cos\frac{2\pi+\theta}{n}+1\right).\left(y^2-2y.\cos\frac{4\pi+\theta}{n}+1\right).\\&\left(y^2-2y.\cos\frac{6\pi+\theta}{n}+1\right)\ldots\left(y^2-2y.\cos\frac{2(n-1)\pi+\theta}{n}+1\right).\end{aligned}\right\}\quad(1)$$

It is obvious that the preceding series is justly terminated, for if we take in other subsequent factors, we would obtain those already found; for instance, we cannot have the factor

$$y^2-2y.\cos\frac{2(n+r)\pi+\theta}{n}+1, \text{ for } \cos\frac{2(n+r)\pi+\theta}{n}=\cos\frac{2r\pi+\theta}{n},$$

one already introduced.

The number of factors obtained is n, and $\therefore$ the number of the roots $2n$, as ought to be the case, the order of the equation being $2n$.

If $\theta=0$, in this developement, the first factor becomes $(y-1)^2$; the second factor becomes the same as the last, since

$$\cos\frac{2\pi}{n}=\cos\frac{2(n-1)\pi}{n}$$

$\therefore$ they unite into the single factor

$$\left(y^2-2y.\cos\frac{2\pi}{n}+1\right)^2;$$

the same happens with respect to the third and second last, $\therefore$ extracting the square root on both sides, we find, if n be even,

$$y^n-1=(y-1).\left(y^2-2y.\cos\frac{2\pi}{n}.+1\right)\ldots\ldots\ldots(y+1) \qquad (2)$$

if n be odd, we obtain

$$y^n-1=(y-1).\left(y^2-2y.\cos\frac{2\pi}{n}+1\right)..\left(y^2-2y.\cos\frac{2.\left(\frac{n}{2}+1\right)\pi}{n}+1\right)(3)$$

If $\theta=\pi$, the left hand member of the equation becomes $(y^n+1)^2$, the first and last factors on the right become the same, as also the second and penultimate, &c., $\therefore$ we have, if n be odd,

$$y^n+1=\left(y^2-2y.\cos\frac{\pi}{n}+1\right).\left(y^2-2y\cos\frac{3\pi}{n}+1\right)\ldots\ldots(y+1) \quad (4)$$

if n be even

$$y^n+1=\left(y^2-2y\cos\frac{\pi}{n}+1\right)\ldots\left(y^2-2y.\cos\frac{(n-1).\pi}{n}+1\right) \qquad (5)$$

8. The equation $x^{2n}-2a^n x^n.\cos\theta+a^n=0$ is decomposable into factors by the same method, for by simply dividing its roots by a, we obtain

$$y^{2n}-2y^n.\cos\theta+1=0.$$

The expression for $y^n \mp 1 = 0$ in factors is announced in the works of Cotes as a property of the circle. It was left by the author amongst his papers, and was saved from perishing by Smith, the editor of Cotes' discoveries.*

The further extension of the Theorem to the equation

$$y^{2n} - 2y^n . \cos \lambda + 1 = 0$$

is due to De Moivre.

9. From the Theorem of Cotes, M. L'Huillier, of Geneva, has deduced very many simple and elegant consequences, some of which we shall now proceed to discuss.

The factors of $x^{2m} + a^{2m}$ are all of the form $x^2 - 2ax . \cos\lambda + a^2$; making $x = a = 1$, they become $2.(1 - \cos\lambda) = 4. \sin^2 \frac{1}{2}\lambda$; and we have $x^{2m} + a^{2m} = 2$. Writing then for λ, its values $\frac{\pi}{2m}, \frac{3\pi}{2m}$, &c. and extracting the square root of each factor, we shall find

$$\sqrt{2} = 2^m . \sin \frac{1}{2m} . \frac{\pi}{2} . \sin \frac{3}{2m} . \frac{\pi}{2} . \sin \frac{5}{2m} . \frac{\pi}{2} \ldots \sin \frac{2m-1}{2m} . \frac{\pi}{2} \qquad (A)$$

The formula $x^{2m+1} + a^{2m+1}$, independant of the trinomial factors, $x^2 - 2ax, \cos\lambda + a^2$, has a real factor of the first degree, viz. $(x+a)$, $\therefore$ we have

$$\left.\begin{array}{l} 1 = 2^m . \sin \dfrac{\pi}{2.(2m+1)} . \sin \dfrac{3\pi}{2.(2m+1)} . \sin \dfrac{5\pi}{2.(2m+1)} \ldots\ldots\ldots \\ \qquad \sin \dfrac{(2m-1)\pi}{2(2m+1)} \end{array}\right\} (B)$$

The formula $x^{2m} - a^{2m}$ has a factor $(x^2 - a^2)$, and $(m-1)$ others of the form $x^2 - 2ax . \cos\lambda + a^2$. Dividing $x^{2m} - a^{2m}$ by $(x^2 - a^2)$, we find

$$x^{2m-2} + x^{2m-4} . a^2 \ldots\ldots\ldots\ldots\ldots a^{2m-2}.$$

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* Revocavi tandem ab interitu Theorema pulcherrimum.—*Cotesii Harm. Mens.* page 113, *prefatio.*

This quote, when $x=a=1$, becomes m; and compared with the product of $(m-1)$ trinomial factors, it gives

$$\sqrt{m}=2^{m-1}. \sin\frac{\pi}{2m}. \sin\frac{2\pi}{2m}. \sin\frac{3\pi}{2m}......\sin\frac{(m-1)\pi}{2m} \qquad (C)$$

If we take the factor $(x-a)$ from the formula $x^{2m+1}-a^{2m+1}$, we shall obtain, on the supposition that $x=a=1$,

$$\sqrt{2m+1}=2^{m}. \sin\left(\frac{2}{2m+1}. \frac{\pi}{2}\right)..........\sin\left(\frac{2m}{2m+1}. \frac{\pi}{2}\right) \qquad (D)$$

From inferences such as these, by the Theory of limits, M. Lhuilier has deduced the decomposition into factors of the quantities $\frac{e^{x}-e^{-x}}{2}$ and $\frac{e^{x}+e^{-x}}{2}$, which we have already otherwise deduced.

CHAP. IV.

ON THE USES AND APPLICATIONS OF TRIGONOMETRY IN THE OTHER SCIENCES.

1. The formulæ of Trigonometry, originally intended for the solution of triangles, have, since the advancement of physical knowledge and the consequent extension of mixed mathematics, been found so extensively and powerfully useful in the other parts of science, that its original object is now become one of the least of its applications. This subsidiary use of our science it is the object of this chapter to elucidate; but no reader who has not extended his views into other parts of mathematics, as well as into the Astronomical and Physical sciences, can form a notion of the advantages resulting from a dexterous and skilful application of Trigonometrical formulæ. We have already seen some instances of their use in the solution of Geometrical Theorems and Problems, and though this application does not exactly harmonize with the

natural, or at least the established order of instruction, still in cases where Geometrical evidence cannot be obtained at all, or not without difficulty, analytical investigation by the formulæ of Trigonometry ought to be applied.

2. We shall introduce as an additional instance to those already given, the method of inscribing a polygon of seventeen sides, which is due to M. Gauss. The side of the polygon of seventeen sides is twice the sine of half the arc it cuts off; let $\varphi = \frac{180^\circ}{17} = \frac{\pi}{17}$, then we shall proceed to determine $2. \sin \frac{\pi}{17}$, the side of the polygon. By Art. 1. Chap. I. Part II., we have

$$\cos\varphi + \cos 3\varphi + \cos 5\varphi + \cos 7\varphi + \cos 9\varphi + \cos 11\varphi + \cos 13\varphi + \cos 15\varphi = \tfrac{1}{2},$$

since $A = \varphi$; $B = 2\varphi$; and $(n-1) = 7$. Dividing the terms that compose this quantity into two sets

$$x = \cos 3\varphi + \cos 5\varphi + \cos 7\varphi + \cos 11\varphi$$

$$y = \cos\varphi + \cos 9\varphi + \cos 13\varphi + \cos 15\varphi$$

we shall have in the first place $x + y = \frac{1}{2}$; next by multiplication and substitution of products of cosines for cosines of simple arcs, we obtain, after reduction

$$xy = 2. (\cos 2\varphi + \cos 4\varphi + \cos 6\varphi \ldots\ldots\ldots\ldots \cos 16\varphi)$$

or, since $\cos 2\varphi = -\cos 15\varphi$; $\cos 4\varphi = -\cos 13\varphi$, &c. we have

$$xy = -2. (\cos 15\varphi + \cos 13\varphi + \ldots\ldots\ldots\ldots \cos\varphi)$$

but as above, the quantity within the parenthesis $= \frac{1}{2}$, $\therefore$ we have $xy = -1$. Hence we obtain

$$x = \frac{1 + \sqrt{17}}{4};\ y = \frac{1 - \sqrt{17}}{4}.$$

Again, dividing x and y, each into two parts, viz.

$x=s+t$ $\qquad$ $y=u+z$

$s=\cos 3\phi+\cos 5\phi$ $\qquad$ $u=\cos\phi+\cos 13\phi$

$t=\cos 7\phi+\cos 11\phi$ $\qquad$ $z=\cos 9\phi+\cos 15\phi$

we shall find $st=-\frac{1}{4}$; $uz=-\frac{1}{4}$, and ∴ we shall be able to find the four quantities s, t, u, z.

Again, knowing that $\cos\phi+\cos 13\phi=u$; and that $s=\cos 3\phi+\cos 5\phi=2.\cos 4\phi.\cos\phi=-2.\cos\phi.\cos 13\phi$, we can eliminate $\cos 13\phi$, and find $\cos\phi$ in terms of s and u. Hence we have $\sin\phi$, ∴ &c.

M. Gauss has demonstrated the following very general Theorem:—

"If the number n is prime, and $(n-1)$ the product of the prime factors $2^{\alpha}\,3^{\beta}\,5^{\gamma}$, &c. the divisions of the circle into n equal parts, may always be reduced to α equations of the second degree, β of the third, γ of the fifth, &c."

3. Trigonometrical formulæ may be successfully applied to facilitate the computation of the roots of Algebraic equations.

1°. In solving quadratic equations. If the quadratic be of the form x^2+px-q the roots are $\frac{p}{2}\tan\theta.\tan\frac{\theta}{2}$ and $-\frac{p}{2}.\tan\theta.\cot\frac{\theta}{2}$, or are equal to $\sqrt{q}.\cot\frac{\theta}{2}$ and $-\sqrt{q}.\tan\frac{\theta}{2}$; according as p or q is eliminated by the equation $\tan^2\theta=\frac{4q}{p^2}$ from the Algebraic expression for the roots $\frac{-p}{2}\pm\sqrt{\frac{p^2}{4}+q}$.

If of the form $x^2-px-q=0$, the roots would be of the same form as the preceding, but with both the signs changed.

If of the form $x^2-px+q=0$, then $x=\frac{p}{2}\left\{1\pm\sqrt{1-\frac{4q}{p^2}}\right\}$, or putting $\frac{4q}{p^2}=\sin^2\theta$, $x=p.\cos\frac{^2\theta}{2}$, and $p.\sin\frac{^2\theta}{2}$.

If of the form $x^2+px+q=0$ the roots are the negative of the preceding.

This method of solution is convenient only when the quantities p and q are decimals, as then it is useful to be able to compute by logarithms. The reader may apply the calculus to the following equations, $x^2+\frac{7}{44}.\,x=\frac{1695}{12716}$, whose roots are $\frac{5}{17}$ and $-\frac{339}{748}$, and $x^2+13{,}56\,x-72{,}31=0$ whose roots are $4{,}02557$, and $-17{,}6556$. Both results may be verified by adding the roots, and seeing that they make the co-efficient of the second term with its sign changed.

2°. The roots of cubic equations may also be rendered easily calculable by similar artifices.

In the cubic equation $x^3+px+q=0$ put $x=z-\frac{p}{3z}$, and the equation becomes $z^6+qz^3=\frac{p^3}{27}$, whence we have

$$z=\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}$$

$$\therefore x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\frac{\frac{1}{3}p}{\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}$$

and multiplying the second term in this value of x above and below by

$$\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}},$$

we find as follows

$$x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

This is Cardan's formula deduced in perhaps the simplest manner, which we shall now proceed to adapt to logarithmic computation. We have evidently

$$x = \sqrt[3]{-\tfrac{1}{2}q\left(1-\sqrt{1+\frac{4p^3}{27q^2}}\right)} + \sqrt[3]{-\tfrac{1}{2}q\left(1+\sqrt{1+\frac{4p^3}{27q^2}}\right)}$$

now let $\tan^2 B=\frac{4p^3}{27q^2}$, then

$$x = \sqrt[3]{-\tfrac{1}{2}q\left(\frac{\cos B-1}{\cos B}\right)} + \sqrt[3]{-\tfrac{1}{2}q.\left(\frac{\cos B+1}{\cos B}\right)}$$

now put $R=2\sqrt{\tfrac{1}{3}p}$, then

$$\tfrac{1}{2}q = \frac{R^3}{8.\tan B}$$

by substitutihg which

$$x=\tfrac{1}{2}.R\left\{\sqrt[3]{\frac{1-\cos B}{\sin B}} - \sqrt[3]{\frac{1+\cos B}{\sin B}}\right\} =$$

$$\tfrac{1}{2}R\{\sqrt[3]{\tan\tfrac{1}{2}B} - \sqrt[3]{\cot\tfrac{1}{2}B}\}$$

now

$$\cot 2A = \frac{\cot A - \tan A}{2}$$

and $\therefore$ if $\tan A=\sqrt[3]{\tan\tfrac{1}{2}B}$, we have

$$x = -R.\cot 2A.$$

Thus by means of three very simple equations we can compute x.

If the equation to solve had been of the form $x^3+px-q=0$, in the solution, $-\tfrac{1}{2}q$ would have become positive, and thus this final result would have been the same as the above, but with a contrary sign.

In the form x^3-px+q, we have in Cardan's formula

$$x=\sqrt[3]{-\tfrac{1}{2}q\,(1-\cos B)} + \sqrt[3]{-\tfrac{1}{2}q\,(1+\cos B)}$$

putting $\frac{4p^3}{27q^2} = \sin^2 B$; which by a contrivance similar to the preceding can be reduced to the form

$$x = -R.\,\text{cosec}\,2A,$$

and thus x be found as before by three equations.

If the equation had been of the form $x^3 - px - q = 0$, this solution would be the same, but with a contrary sign.

In these two latter cases, should it happen that $\frac{p^3}{27} > \frac{q^2}{4}$ the expression derived from Cardan's formula is of an imaginary form, although $\frac{p^3}{27} > \frac{q^2}{4}$ is the criterion that the three roots are real. This, called by writers on Algebra the *irreducible case*, would, according to the preceding methods, involve us in the developement of imaginary results. There are, however, two very simple Trigonometrical formulæ which apply to this case.

1°. For the case $x^3 - px + q = 0$, we have

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

whence

$$\sin^3 A - \frac{3R^2}{4}.\sin A + \frac{R^2}{4}.\sin 3A = 0,$$

with the radius R. By comparing this with the given equation, we have

$$p = \tfrac{3}{4}.R^2,\ q = \frac{R^2}{4}.\sin 3A,$$

hence we have $3A$, and $\therefore A$, and $\therefore \sin A$ or x.

2°. For the case $x^3 - px - q = 0$, we have

$$\cos 3A = 4.\cos^3 A - 3\cos A,$$

$\therefore \cos^3 A - \frac{3R^2}{4}.\cos A - \frac{R^2}{4}.\cos 3A = 0$, which by comparison gives us $\cos A$ or x, as before.

4. Several interesting exemplifications of the utility of Trigonometrical formulæ may be found in the details that have been published of the Trigonometrical surveys; although indeed these details are more valuable as exemplifications of the *practical* than of the merely Theoretical parts of our science.

1°. From the observed angular distance of two objects not in the horizon of the observer, it is requisite to be able to compute the angular distance of the same objects referred to this horizon.

Let AOB (Fig. 21.) be the observed angle, then this angle on the plane of the horizon, viz. aob, is equal to the angle Z at the zenith; but the angle Z may be found from knowing ZA, ZB, and AB, by any one of the formulæ in the First Case of oblique spherical triangles.—*Vide page* 51.

2°. Any one of these formulæ is sufficiently short for the purposes of a single observation; still when several hundred such are to be computed, it is necessary to be furnished with an expeditious formula for the purpose.

Let the observed angle $= a$, and the angle required $= (a+x)$; then

$$\cos(a+x)=\frac{\cos a-\sin\beta.\sin\gamma}{\cos\beta.\cos\gamma}$$

where β and γ denote the altitudes of the stations; now since β and γ are in general very small, we may write

$\sin\beta.\sin\gamma=\beta\gamma$, $\cos\beta.\cos\gamma=1-\frac{\beta^2}{2}-\frac{\gamma^2}{2}$, rejecting the fourth and higher powers of β and γ. Hence we have

$$\cos(a+x)=(1+\tfrac{1}{2}\beta^2+\tfrac{1}{2}\gamma^2)\cos a-\beta.\gamma;$$

but

$$\cos(a+x)=\cos a-x.\sin a,$$

rejecting second and higher powers of x,

$$\therefore x=\frac{\beta\gamma-\frac{1}{2}(\beta^2+\gamma^2).\cos a}{\sin a}.$$

Put $\frac{1}{2}(\beta+\gamma)=p$, $\frac{1}{2}(\beta-\gamma)=q$, and we have under a simpler form,

$$x=p^2.\left(\frac{1-\cos a}{\sin a}\right)-q^2\left(\frac{1+\cos a}{\sin a}\right)=p^2.\tan\tfrac{1}{2}a-q^2.\cot\tfrac{1}{2}a.$$

This is the formula for the purpose that has been given by Legendre, *Mem. Acad.* 1787, page 354.—*Trigonometry*, page 352, *Brewster's Translation*. Delambre has given a series for x, with the first term of which the above expression nearly agrees.—*Connoissance des Tems*, 1793.

Legendre remarks, that where β and γ amount to more than two or three degrees, it would be better to make use of the general method.

3°. If all the angles be thus reduced to horizontal angles, they become angles of a spherical triangle on the surface of the sphære; as such, the sum of the three $>$ 180°. If we had then any Theoretical means of ascertaining this excess, we could thereby judge of the accuracy of observation. The Theorem of Albert Girard, demonstrated at page (36) of this treatise, has been very elegantly applied to this purpose by General Roy, in the *Phil. Trans.* for 1790. His rule is as follows:—

"*From the logarithm of the area of the triangle, taken as a plane one in feet, subtract the constant logarithm* 9.3267737, *and the remainder is the logarithm of the excess above* 180° *in seconds, nearly.*"

This rule we shall proceed to investigate. By Albert Girard's Theorem, proved in page 36, we have

$$\Sigma=r^2\,(A+B+C-180°)=r.\,(A+B+C-180°)$$

if A, B, C, and 180° be taken in the circle whose radius is r; $\therefore$ denoting the excess by ϵ, we have

$$\epsilon=\frac{180.\,\Sigma}{\pi r^2},$$

or in seconds,

$$\epsilon''=\frac{180.\,60.\,60}{\pi r^2}.\,\Sigma;$$

now on the earth's surface, 1° (taking a mean measurement) corresponds to (60859.1) × 6 feet; therefore since an arc (=radius)= $\frac{360}{2\pi}$, we have the earth's radius in feet $= \frac{360}{2\pi} \times (60859.1) \times 6$.
Hence the excess in seconds

$$= x \times \frac{2\pi}{360} \times \frac{60 \times 60}{6^2 \times (63859.1)^2} = x. \frac{2\pi . 10}{36 \times (60859.1)^2},$$

∴ log. excess,

$$= \log. x - \{2 (\log. 6 + \log. 60859.1) - \log. 2\pi . 10\}$$

$$= \log. x - 9.3267737.$$

Having in this manner examined the accuracy of observation, and corrected accordingly, the next business is to calculate the triangles by the rules of spherical Trigonometry. Thus it was that Boscovich proceeded, but this method does not give its results very expeditiously. Legendre has remedied this inconvenience, by combining sufficient exactness with conciseness, in the following Theorem:—

" *A spherical triangle being proposed, of which the sides are small compared with the radius of the sphere, if from each of the angles one-third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, the sides of which are equal in length to those of the proposed spherical triangle.*"

To determine this correction we have

$$\cos A = \frac{\cos a - \cos b . \cos c}{\sin b . \sin c}$$

Let α, β, γ, be the arcs in a circle whose radius is r, that correspond to a, b, c, then $a = \frac{\alpha}{r}$, $b = \frac{\beta}{r}$, $c = \frac{\gamma}{r}$; introducing these values in seiies (3) and (4), page 104, and substituting from these series in cos A, we obtain

$$\cos A = \frac{\frac{\beta^2+\gamma^2-\alpha^2}{2\,r^2}+\frac{\alpha^4-\beta^4-\gamma^4}{2.\,3.\,4.\,r^4}-\frac{\beta^2\,\gamma^2}{4\,r^4}}{\frac{\beta\gamma}{r^2}.\left(1-\frac{\beta^2+\gamma^2}{2.\,3.\,r^2}\right)}$$

multiplying the numerator and denominator of the fraction by r^2, and multiplying by $1+\frac{\beta^2+\gamma^2}{2.\,3.\,r^2}$, instead of dividing by $1-\frac{\beta^2+\gamma^2}{2.\,3.\,r^2}$, we have, neglecting such terms as are divided by powers above r^2,

$$\cos A=\frac{\beta^2+\gamma^2-\alpha^2}{2\,\beta\gamma}+\frac{\alpha^4-\beta^4-\gamma^4}{2.3.4.\beta.\gamma r^2}-\frac{\beta^2\,\gamma^2}{4\beta\,\gamma.\,r^2}+\frac{(\beta^2+\gamma^2-\alpha^2).(\beta^2+\gamma^2)}{3.\,4.\,\beta\,\gamma r^2}$$

$$=\frac{\beta^2+\gamma^2-\alpha^2}{2\,\beta\,\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\,\alpha^2\,\beta^2-2\,\alpha^2\,\gamma^2-2\,\beta^2\,\gamma^2}{2.\,3.\,4.\,\beta\,\gamma.\,r^2}$$

By making $\frac{1}{r}=0$ the angle A becomes the angle opposite the side α in the rectilinear triangle, of which α, β, γ, are the sides. Let us denote the angle in this state by A', then we have

$$\cos A' = \frac{\beta^2+\gamma^2-\alpha^2}{2\,\beta.\,\gamma},$$

and hence

$$\sin^2 A' = \frac{2\,\alpha^2\,\beta^2+2\,\alpha^2\,\gamma^2+2\,\beta^2\,\gamma^2-\alpha^4-\beta^4-\gamma^4}{4\,\beta^2\,\gamma^2}.$$

Substituting these values in the expression above given for $\cos A$, we have

$$\cos A=\cos A'-\frac{\beta\,\gamma.\,\sin^2 A'}{2.\,3.\,r^2}.$$

But the area of the rectilinear triangle

$$=\frac{\beta\,\gamma.\,\sin A'}{2},$$

$\therefore$ if θ be put to denote this area, we have

$$\cos A = \cos A' - \frac{\theta . \sin A'}{3 r^2}.$$

Hence we can obtain, to quantities of the order $\frac{1}{r^4}$, nearly,

$$A = A' + \frac{\theta}{3 r^2}$$

By similar reasoning we obtain

$$B = B' + \frac{\theta}{3 r^2}$$

$$C = C' + \frac{\theta}{3 r^2};$$

but

$$\frac{\theta}{r^2} = A + B + C - 180°,$$

∴ the Theorem as announced is true.

The substance of this proof is to be found in Lagrange's Memoir, Journal de l'Ecole Polytechnique, Vol. II. page 270.

The triangles solved according to the preceding methods are spherical triangles, and an arc of the meridian if computed from such would be an arc of a great circle. These methods, however, are not practically necessary, as Delambre, the most accurate of computists, and Colonel Mudge, in the Survey of 1803, have resolved triangles, not as spherical, but as rectilinear triangles, and accordingly do not consider the arc of the meridian as a curve, but as formed of the chords of curves. The oblique angles are first to be reduced to horizontal angles, as in the preceding methods, and then from the spherical angles so determined, we have to find the angles of the chords. For this purpose the following seems the most simple mode of proceeding.

In (Fig. 22.) let BAC be the angle of the chords; complete the arcs AB, AC into semicircles, and join Aa; from the centre D draw Db, Dc parallel respectively to AB, AC; then the angle

bDc (=arc bc)=angle BAC; and the arcs Ca, Ba, are obviously bisected. We have then

$$\cos bc = \cos ba . \cos ac + \sin ba . \sin ac . \cos bac;$$

but ba, ca, are complements to the halves of the arcs AB, AC; and the angle bac=the spherical angle A, ∴ we have

$$\cos BAC = \sin \frac{AC}{2} . \sin \frac{AB}{2} + \cos \frac{AC}{2} . \cos \frac{AB}{2} . \cos A.$$

The test of accuracy, according to this method of proceeding, is, that the sum of the three angles of the chords found from the spherical angles ought to be equal to 180°. The calculation is then effected by the rules of plane Trigonometry, and an arc of the meridian determined by this means is composed of the sides of an irregular polygon.

Thus we see there are three methods in all:—either accurate computation by the rules of spherical Trigonometry; approximate computation by the same; or computation by the rules of plane Trigonometry. Delambre computed by all three the whole series of triangles from Dunkirk to Barcelona.

5. We might introduce a great variety of instances from plain Astronomy, as in this science particularly, Trigonometry is of most essential and indispensable use. We shall, however, restrict ourselves to the detail of a few of the mathematical difficulties of this interesting science.

1. *Required to investigate a formula for the apparent motion of a planet as seen from the earth.*

Taking the case of the inferior planet in (Fig. 23.), supposing the planet to have moved from P to P, and the earth from T to T in the same time, then the whole apparent motion of the planet as seen from T, is the sum or difference of the angles PTP and TPT. Letting fall Po, Tn, perpendiculars on PT, TP, respectively, the angle $PTP = \frac{Po}{PT}$, and $TPT = \frac{Tn}{PT}$. Let TT, PP, the velocities of the earth and planet, be denoted by the cha-

racters v, v'; the angles STP, SPT, by ι and ι'; ST, SP, by r and r' respectively; and TP by d. Then the apparent motion

$$M=\frac{v.\cos\iota\pm v'.\cos\iota'}{d},$$

but

$$\frac{v}{v'}=\frac{\sqrt{r'}}{\sqrt{r}},\ \therefore M=\frac{v.\sqrt{r'}.\cos\iota\pm v.\sqrt{r}.\cos\iota'}{d.\sqrt{r'}}.$$

This expression becoming negative denotes that the planet is retrograde. And the planet is stationary when $\sqrt{r'}.\cos\iota=\sqrt{r}.\cos\iota'$. From this equation, and the equation $r.\sin\iota=r'.\sin\iota'$, we have, when the planet is stationary,

$$\sin^2\iota=\frac{r'^2}{r'^2+rr'+r^2},$$

the same as Dr. Brinkley's expression, page 299.

2. *Required to compute an expression for the parallax in altitude* p, *in terms of the horizontal parallax* P, *and true zenith distance* D.

We have obviously $\sin p=\sin P.\sin(D+p)$; hence

$$\tan p=\frac{\sin P.\sin D}{1-\sin P.\cos D};$$

by series (6), page 106, we have

$$p=\tan p-\tfrac{1}{3}.\tan^3 p+\&c.=\frac{\sin P.\sin D}{1-\sin P.\cos D}-\frac{1}{3}\frac{\sin^3 P.\sin^3 D}{(1-\sin P.\cos D)^3}+\&c.$$

but

$$\frac{\sin P.\sin D}{1-\sin P.\cos D}=\sin P.\sin D+\sin^2 P.\sin D.\cos D+\sin^3 P.\sin D.\cos^2 D$$

and

$$\frac{\sin^3 P.\ \sin^3 D}{(1-\sin P.\ \cos D)^3}=\sin^3 P.\ \sin^3 D+\&c.$$

Hence, collecting the terms, we have

$$p=\sin P.\ \sin D+\sin^2 P.\ \sin D.\ \cos D+\left(\frac{3\sin D-4\sin^3 D}{3}\right).\ \sin^3 P$$

$$=\sin D.\ \sin P+\tfrac{1}{2}\sin 2D.\ \sin^2 P+\tfrac{1}{3}.\ \sin 3D.\ \sin^3 P,$$

which it is evident is but a part of a series obeying the same law. —*Vide* Woodhouse's Astronomy, Vol. I. page 324;—Brinkley, page 319, Appendix.

3. *To investigate a formula for the reduction of the ecliptic to the equator.*

Let A be the sun's right ascension, and L the sun's longitude, then the equation is to investigate a series for $L-A$. We have

$$\tan(L-A)=\frac{\tan L-\tan A}{1+\tan L.\ \tan A};$$

but by Napier's rules, denoting the obliquity of the ecliptic by w, we have

$$\tan A=\cos w.\ \tan L.$$

Hence we have

$$\tan(L-A)=\frac{\tan L\,(1-\cos w)}{1+\cos w.\ \tan^2 L},$$

but

$$\tan L=\frac{\sin 2L}{2.\cos^2 L},$$

$\therefore \tan(L-A)=$

$$\frac{\sin 2L.\ (1-\cos w)}{2\cos^2 L+2\cos w.\ \sin^2 L}=\frac{\sin 2L.\ (1-\cos w)}{2-2\sin^2 L\,(1-\cos w)}$$

$$=\frac{\sin 2L.(1-\cos w)}{1+\cos w+(1-2\sin^2 L).(1-\cos w)};$$

or since $\frac{1-\cos w}{1+\cos w}=\tan^2\frac{w}{2}=t^2$, we have

$$\tan(L-A)=\frac{t^2.\sin 2L}{1+t^2.\cos 2L};$$

whence, by series (6), page 106,

$$L-A=\frac{t^2\sin 2L}{1+t^2\cos 2L}-\tfrac{1}{3}.\frac{t^6.\sin^3 2L}{(1+t^2\cos 2L)^3}.+\&c.;$$

or treating the terms as in the preceding question, we find

$$L-A=t^2.\sin 2L-\tfrac{1}{2}t^4.\sin 4L+\tfrac{1}{3}t^6.\sin 6L-\&c.$$

This series is arrived at by a most laboured process in Woodhouse's Astronomy, Vol. I. page 501. The series may be written as follows; since $\sin 2''=2.\sin 1''$, &c. we have

$$L-A=\tan^2\frac{w}{2}.\frac{\sin 2L}{\sin 1''}-\tan^4\frac{w}{2}.\frac{\sin 4L}{\sin 2''}+\&c.$$

4. *Required an investigation of the equations of the solar Theory.*

In (Fig. 24.) the time t of moving from the perihelion B to the point P, is to the periodic time P, as the area BEB to the area of the ellipse; or by the properties of the conic sections, as the area DEB to the area of the circle,

$$\therefore t=\frac{P}{\pi a^2}.(DCB-DCE)=\frac{P}{2\pi a^2}(a^2u-ac.\sin u)=\frac{P}{2\pi}(u-e.\sin u)$$

hence, if we put $\frac{P}{2\pi}=\frac{1}{n}$, we have

$$nt=u-e.\sin u \qquad (a)$$

The quantity u is called the eccentric anomaly, and nt the mean anomaly; the next object is to investigate an equation connecting the eccentric with the true anomaly, which we shall denote by the character v. For this purpose we shall investigate and equate two values of the *radius vector* r.

By the properties of the conic sections, we have in terms of the true anomaly,

$$r=\frac{a\,(1-e^2)}{1+e\cos v}. \qquad (b)$$

Again we have

$$DE^2-PE^2=DN^2-PN^2=(a^2-x^2)-\frac{b^2}{a^2}.\,(a^2-x^2)=$$

$$\frac{c^2.\,(a^2-x^2)}{a^2}=c^2.\,\sin^2 u;$$

but

$$DE^2-DT^2=c^2.\,\sin^2 u, \qquad \therefore DT=EP,$$

but

$$DT=a-c.\cos u=a\,(1-e\cos u), \qquad \therefore r=a.\,(1-e\cos u) \qquad (c)$$

Equating (b) and (c), we have

$$(1-e^2)=(1+e.\cos v).\,(1-e.\cos u),$$

whence

$$\frac{\cos u-e}{1-e\cos u}=\cos v;$$

but

$$\tan\frac{v}{2}=\sqrt{\frac{1-\cos v}{1+\cos v}}=\sqrt{\frac{(1+e).(1-\cos u)}{(1-e).(1+\cos u)}}=\sqrt{\frac{1+e}{1-e}}.\tan\frac{u}{2} \qquad (d)$$

The equations so far found, analytically resolve the problem;

but it is necessary to be able to express v in terms of nt directly. This operation cannot be performed without the aid of the Theorems of Laplace and Lagrange.

These Theorems, from their very great power and generality in the developement of functions, we shall accordingly introduce for their own sakes, and exemplify in the resolution of the problem at present before us. We shall commence with the general Theorem of Laplace, and then deduce as a particular case of it, that first arrived at by Lagrange.

The object is to turn into a series, ascending by the powers of x, the very general function $u=\psi(y)$, where y is a function of the independant variable x, expressed by the equation $y=F(a+x\phi(y))$, where F and ϕ are characteristics of any given functions.

By the series of M^c. Laurin,

$$u=U+\frac{d.\psi(y)}{dx}\cdot\frac{x}{1}+\frac{d^2.\psi(y)}{dx^2}\cdot\frac{x^2}{1.2}+\frac{d^3\psi(y)}{dx^3}\cdot\frac{x^3}{1.2.3.}+\frac{d^4\psi(y)}{dx^4}\cdot\frac{x^4}{1.2.3.4}, \text{ \&c.}$$

it being understood that in these several diff. coeff. $x=0$.

The difficulty now remaining is to introduce for the diff. coeff. on the hypothesis of x being the variable; equivalents for them that will involve differential coeff. on the hypothesis that a is the variable.

Differentiating the second equation on the separate hyp. of x and a being variable, we have

$$\frac{dy}{dx}=F'(a+x\phi(y))\left(\phi y+x.\phi'(y).\frac{dy}{dx}\right)$$

$$\frac{dy}{da}=F'(a+x\phi(y))\left(1+x.\phi'(y).\frac{dy}{da}\right)$$

This is true, for

$$dy=F'(a+x\phi(y))\times d.(a+x.\phi(y)),$$

where the character d denotes the complete differential in both cases, but the complete diff. = sum of partial diff.

$$\therefore dy = F'(a+x\varphi(y))\left\{\left(\varphi y + x.\varphi'(y).\frac{dy}{dx}\right)dx + \left(1+x.\varphi'(y).\frac{dy}{da}\right)da\right\}$$

and

$$dy = \frac{dy}{dx}.\,dx + \frac{dy}{da}.\,da,$$

but of course the partial differentials are separately equal, and by what is here said, they have the same co-efficient if any. The same reasoning accounts for $\varphi'(y)$ being the same in both.

Eliminating $F'(a+x\varphi(y))$, we find

$$\frac{dy}{dx} - \varphi(y).\frac{dy}{da} = 0.$$

u or $\psi(y)$ being merely a function of y, we have on the same distinct hypotheses

$$\frac{du}{dx} = \psi'(y).\frac{dy}{dx};\ \frac{du}{da} = \psi'(y).\frac{dy}{da},\ \therefore \frac{du}{dx}.\frac{dy}{da} = \frac{du}{da}.\frac{dy}{dx},$$

but

$$\frac{dy}{dx} = \varphi(y).\frac{dy}{da},\ \therefore \frac{du}{dx} = \varphi(y)\frac{du}{da}.$$

This is a value for the first differential co-efficient.

Differentiate this value, supposing $\varphi(y).\frac{du}{da} = u'$, it being a function of y, then

$$\frac{d^2u}{dx^2} = \frac{d^2u'}{dx\,da} = \frac{d.\frac{du'}{dx}}{da};$$

but

$$\frac{du'}{dx}=\phi(y).\frac{du'}{da},$$

for in consequence of u' being a function of y, we have

$$\frac{du'}{dx}.\frac{dy}{da}=\frac{du'}{da}.\frac{dy}{dx},$$

and substituting its value for $\frac{dy}{dx}$, we have

$$\frac{du'}{dx}=z.\frac{du'}{da},$$

where $z=\phi(y)$. Writing then in the second value of $\frac{d^2u}{dx}$, this its value for $\frac{du'}{dx}$, and then for $\frac{du'}{da}$ its value $z.\frac{du}{da}$, we obtain

$$\frac{d^2u}{dx^2}=\frac{d.z^2.\frac{du}{da}}{da}.$$

This is a value for the second diff. coeff.

Differentiating this equation relative to x again,

$$\frac{d^3u}{dx^3}=\frac{d^2.z^2.\frac{du}{da}}{dx.da},$$

making $z^2.\frac{du}{da}=\frac{du''}{da}$, and changing the order of the operations, we find

$$\frac{d^3u}{dx^3}=\frac{d^2.\frac{du''}{dx}}{da^2},$$

but we have also

$$\frac{du''}{dx} = z.\frac{du''}{da},$$

as before, and

$$\therefore \frac{du''}{dx} = z^3.\frac{du}{da} \therefore \frac{d^3 u}{dx^3} = \frac{d^2.z^3\frac{du}{da}}{da^2}.$$

This is the value of the third diff. coeff.

And in general if

$$\frac{d^{n-1} u}{dx^{n-1}} = \frac{d^{n-2}.z^{n-1}.\frac{du}{da}}{da^{n-2}},$$

we can prove it true of the next succeeding. For making

$$z^{n-1}.\frac{du}{da} = \frac{d^n u^{''\cdots(n-1)}}{da^{n-1}.dx},$$

$(n-1)$ designating not a power, but the number of accents that u carries, we find

$$\frac{d^n u}{dx^n} = \frac{d^n.u^{''\cdots(n-1)}}{dx.da^{n-1}} = \frac{d^n.u^{'''\cdots(n-1)}}{da^{n-1}.dx},$$

and because of

$$\frac{du^{'''\cdots(n-1)}}{dx} = z.\frac{du^{'''\cdots(n-1)}}{da} = z^n.\frac{du}{da}$$

we have

$$\frac{d^n.u}{dx^n} = \frac{d^{n-1}.z^n.\frac{du}{da}}{da^{n-1}}.$$

Thus we have proved that if the law of differential coeff. be true of the $(n-1)^{th}$ it is true of the n^{th}, but it is true of the first, second, third, &c. and $\therefore$ of every co-efficient. Putting all together, we have

$$\left.\begin{array}{l} u = U + z.\dfrac{du}{da}.\dfrac{x}{1} + \dfrac{d.z^2.\dfrac{du}{da}}{da}.\dfrac{x^2}{1.2} + \dfrac{d^2.z^3.\dfrac{du}{da}}{da^2}.\dfrac{x^3}{1.2.3}\ldots\ldots \\ \dfrac{d^{n-1}.z^n.\dfrac{du}{da}}{da^{n-1}}.\dfrac{x^n}{1.2.3\ldots\ldots n} \end{array}\right\} \quad (1)$$

U denoting what u becomes when $x=0$. The same supposition is of course to be made on $y=F(a+x\varphi(y))$, which gives

$$y=F(a), \frac{dy}{da}=F'(a), \text{\&c.}$$

as values to be used in z or $\varphi(y)$, and in $\dfrac{du}{da}$.

This Theorem is due to M. Laplace, who gives a demonstration of it differing but little from the above.—*Vide Mec. Cal.* tom. I. page 170;—Lacroix *Calcul. Diff.* tom. I. page 279.

In the particular case where $y=a+x\varphi(y)$, we have

$$y=a; \frac{dy}{da}=1;$$

z and $\dfrac{du}{da}$ become respectively $\psi(a)$, $\varphi(a)$ and $\psi'(a)$, consequently u arranged according to the powers of x will become

$$\left.\begin{array}{l} u=\psi(a)+\psi'(a)\varphi(a).\dfrac{x}{1}+\dfrac{d.\psi'(a)\varphi(a)^2}{da^2}.\dfrac{x^2}{1.2}\ldots\ldots\ldots\ldots \\ \dfrac{d^{n-1}.\psi'(a)\varphi(a)^n}{da^{n-1}}.\dfrac{x^n}{1.2\ldots\ldots n} \end{array}\right\} \quad (2)$$

If $x=1$, we have

$$y=a+\varphi(y)$$

and the series becomes

$$\left.\begin{array}{l}\psi(y)=\psi(a)+\psi'(a)\,\phi(a)+\frac{1}{1.2}\cdot\frac{d.\,\psi'(a)\,\phi(a)^2}{da}+\\ \frac{1}{1.2.3}\cdot\frac{d^2.\,\psi'(a)\,\phi(a)^3}{da^2}+\&c.\end{array}\right\}\quad(3)$$

This last form of the series is that which M. Lagrange arrived at by induction from the roots of Algebraic equations.—*Vide Note* xi, *Theorie des equations Numeriques.*

As the introduction of these Theorems has been, at least in a secondary sense, with a view to shewing the utility of formulæ, a more elegant and appropriate illustration of them cannot possibly be afforded than is given by Laplace himself, in his application to the solar Theory.

First, from the equation $nt=u-e\sin u$, where nt and u are the mean and eccentric anomalies, and e the eccentricity; let us seek the developement of u in a series ascending by the powers of e. We have $u=nt+e.\sin u$, to find u. This is an instance of the second of the preceding series in which $u=y$, ∴ we have

$$\psi(u)=\psi(nt)+e.\,\psi'(nt).\,\sin nt+\frac{e^2}{1.2}\cdot\frac{d.\,\{\psi'(nt).\sin^2 nt\}}{ndt}+$$

$$\frac{e^3}{1.2.3}\cdot\frac{d^2\{\psi'(nt).\,\sin^3 nt\}}{n^2\,dt^2}+\frac{e^4}{1.2.3.4}\cdot\frac{d^3\{\psi'(nt).\,\sin^4 nt\}}{n^3\,dt^3}+\&c.$$

But as $\psi(u)=u$, we have

$$\psi'.\,nt=1,$$

∴ this series becomes

$$u=nt+e.\sin nt-\frac{e^2}{1.2.}\cdot\frac{1}{2}\,\frac{d.\,\{\cos 2\,nt-1\}}{ndt}-$$

$$\frac{e^3}{1.2.3}\cdot\frac{1}{2^2}\cdot\frac{d^2\{\sin 3\,nt-3.\sin nt\}}{n^2\,dt^2}+$$

$$\frac{e^4}{1.2.3.4}\cdot\frac{1}{2^3}\cdot\frac{d^3\{\cos 4\,nt-4.\cos 2\,nt+3\}}{n^3\,dt^3}+$$

$$\frac{e^5}{1.2.3.4.5}\cdot\frac{1}{2^4}\cdot\frac{d^4\{\sin 5nt - 5\sin 3nt + 10.\sin nt\}}{n^4\,dt^4} - \&c.$$

These reductions are instantly obvious from Chap. II. Part I. page 13, of this treatise.

Performing the differentiations denoted by the character d, and dividing by the differentials $n\,dt$, $n^2\,dt^2$, &c. wc obtain the following series:

$$\left.\begin{array}{l} u = nt + e\sin nt + \dfrac{e^2}{1.2.2}\cdot 2\sin 2nt + \\ \dfrac{e^3}{1.2.3.2^2}\cdot\{3^2.\sin 3nt - 3.\sin nt\} + \\ \dfrac{e^4}{1.2.3.4.2^3}\{4^3.\sin 4nt + 4.2^3.\sin 2nt\} + \\ \dfrac{e^5}{1.2.3.4.5.2^4}\{5^4.\sin 5nt - 5.3^4.\sin 3nt + 10.\sin nt\} + \&c. \end{array}\right\}(A)$$

This series, in consequence of the smallness of the quantity e, is very convergent for the planets. We may obtain r from the equation

$$r = a(1 - e\cos u),$$

in which r is the radius vector of the planet's orbit, since u is known from series (A). Or *a priori*, as follows.

We have

$$\psi(u) = 1 - e\cos u,$$

or using for u its value in terms of nt, we have

$$\psi(nt + e.\sin u) = -e.\cos u, \therefore \frac{d.(\psi(nt + e.\sin u))}{d.nt} = -e.\sin u,$$

$\therefore$ supposing $e = 0$, we have

$$\psi'(nt) = e.\sin nt,$$

hence we have

$$1 - e\cos u = 1 - e\cos nt + e^2 . \sin^2 nt + \frac{e^3}{1.2} . \frac{d.\sin^3 nt}{ndt} + \frac{e^4}{1.2.3} . \frac{d^2.\sin^4 nt}{n^2dt^2} + \&c.$$

Differentiating as required, and introducing the expressions for the powers of the sine, we have

$$\left.\begin{aligned}
\frac{r}{a} = 1 + \frac{e^2}{2} - e.\cos nt - \frac{e^2}{2}.\cos 2\,nt \\
-\frac{e^3}{1.2.2^2}\{3\cos 3\,nt - 3\cos nt\} \\
-\frac{e^4}{1.2.3.2^3}\{4^2.\cos 4\,nt - 4.\,2^2.\cos 2\,nt\} \\
-\frac{e^5}{1.2.3.4.2^4}\{5^3.\cos 5\,nt - 5.\,3^3.\cos 3\,nt + 10\cos nt\} \\
-\frac{e^6}{1.2.3.4.5.2^5}\{6^4.\cos 6nt - 6.4^4.\cos 4\,nt + 15.2^4.\cos 2nt\} \\
\&c. \qquad \&c. \qquad \&c.
\end{aligned}\right\}(B)$$

It remains to discover the value of the true in terms of the mean anomaly. We have from the solar Theory,

$$\tan \tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}}.\tan \tfrac{1}{2}u;$$

using for $\tan \frac{1}{2}v$, $\tan \frac{1}{2}u$, their exponential values given in page 105 of this work, we have

$$\frac{\varepsilon^{v\sqrt{-1}} - 1}{\varepsilon^{v\sqrt{-1}} + 1} = \sqrt{\frac{1+e}{1-e}}.\frac{\varepsilon^{u\sqrt{-1}} - 1}{\varepsilon^{u\sqrt{-1}} + 1},$$

or putting $\lambda = \frac{e}{1+\sqrt{1-e^2}}$, we have

$$\varepsilon^{v\sqrt{-1}} = \varepsilon^{u\sqrt{-1}}. \left\{\frac{1-\lambda\,\varepsilon^{-u\sqrt{-1}}}{1-\lambda\,\varepsilon^{u\sqrt{-1}}}\right\},$$

or taking the logarithms, we have

$$v=u+\frac{l\,(1-\lambda\,\varepsilon^{-u\sqrt{-1}})-l.\,(1-\lambda.\,\varepsilon^{u\sqrt{-1}})}{\sqrt{-1}}.$$

Expanding the numerator of the fractional part in two logarithmic series, subtracting those series, and using the sine for its exponential value, we have

$$\left.\begin{aligned} v=u+2\,\lambda.\,\sin u+\frac{2\,\lambda^2}{2}.\,\sin 2\,u+\frac{2\,\lambda^3}{3}.\,\sin 3\,u+\\ \frac{2\,\lambda^4}{4}.\,\sin 4u+\frac{2\,\lambda^5}{5}.\,\sin 5\,u+\&c.\end{aligned}\right\}(C)$$

As our object is to obtain a series for v in terms of the sines of $n\,t$ and its multiples, and arranged so as to involve powers of e to a known extent, we have to expand λ, and to substitute for u, $\sin u$, &c. from series (A).

Take $y=2-\frac{e^2}{y}$, then let us proceed to find $\frac{1}{y^n}$ by series (3). We have

$$\psi\,(y)=\frac{1}{y^n},\quad \therefore\,\psi'\,(y)=-\frac{n}{y^{n+1}},$$

and

$$\varphi\,(y)=-\frac{e^2}{y},\quad \therefore\,\psi\,(a)=\frac{1}{2^n},\ \psi'\,(a)=-\frac{n}{2^{n+1}},\ \varphi\,(a)=-\frac{e^2}{2},$$

putting all which together, we find

$$\frac{1}{y^n}=\frac{1}{2^n}+\frac{n.}{2^{n+2}}.\,\frac{e^2}{1}+\frac{n.\,n+3}{1.\,2}.\,\frac{e^4}{2^{n+4}}+\frac{n.\,n+3.\,n+5}{1.\,2.\,3}.\,\frac{e^6}{2^{n+6}},$$

now since y has been assumed for $1+\sqrt{1-e^2}$, we have

$$\lambda^n=\frac{e^n}{y^n}$$

$$\therefore \lambda^n = \frac{e^n}{2^n}\left\{1 + n.\left(\frac{e}{2}\right)^2 + \frac{n.\,n+3}{1.\,2}.\left(\frac{e}{2}\right)^4 + \frac{n.\,n+3.\,n+5}{1.\,2.\,3}.\left(\frac{e}{2}\right)^6 + \&c.\right\}$$

Having supposed in this series n successively 1, 2, 3, &c. let us introduce the values so found for λ, λ^2, λ^3, &c. into series (C), let us substitute for $\sin u$, $\sin 2u$, $\sin 3u$, &c. their values in terms of u, and then replace u by its value had from series (A), and we obtain the following series,

$$\left.\begin{array}{l} v = nt + \{2\,e - \frac{1}{4}\,e^3 + \frac{5}{96}\,e^5\}\sin nt + \{\frac{5}{4}.\,e^2 - \frac{11}{24}.\,e^4 \\ + \frac{17}{192}.\,e^6\}\sin 2\,nt + \{\frac{13}{12}.\,e^3 - \frac{43}{64}\,e^5\}\sin 3\,nt + \&c. \end{array}\right\} (D)$$

The angles v and nt are here understood to be counted from the perihelion; but if we wish to count from the aphelion, we need only make e negative in the preceding expressions for r and v.

On this subject, see Laplace *Mem. Acad.* 1777, and *Mec. Celeste*, page 170, &c. Vol. I.; Lagrange *Theorie des Fonctions Anal.* page 101, &c.

Woodhouse applies the Theorem of Lagrange to deduce expressions for the sine and cosine of a multiple arc, which expressions, he remarks, admit of no simple proofs.

6. We shall now proceed to exemplify the utility of our science by some few applications of it to the integral calculus.

To integrate the differential expressions $dz.\cos z.\ \sin z.\ dz$; $\cos^3 z$; $dz.\sin^3 z$; $dz.\sin mz.\sin nz$; $dz.\sin mz.\cos nz$.

1°. $\int dz.\cos z.\sin z = \frac{1}{4}\int d.\,2z.\ \sin 2z = -\frac{1}{4}.\cos 2z + \text{constant}.$

2°. $\int dz\cos^3 z = \int dz.\,(\frac{1}{4}.\cos 3z + \frac{3}{4}\cos z) = \frac{1}{3.\,4}.\ \sin 3z + \frac{3}{4}.\sin z + \text{constant}.$

3°. $\int dz.\sin^3 z = \int dz.\,(-\frac{1}{4}\sin 3z + \frac{3}{4}\sin z) = \frac{1}{3.\,4}.\ \cos 3z - \frac{3}{4}\cos z + \text{constant}.$

These integrations may be applied to the general forms

$$\int \cos^n z.\ dz; \int \sin^n z.\ dz,$$

by means of the series that have been given in Articles 29 and 30 of Chap. III. Part II.

4°. $\int dz.\ \sin mz.\ \sin nz = \frac{1}{2}\int dz\,(\cos(m-n)z - \cos(m+n)z)$

$$= \frac{\sin(m-n)z}{2.(m-n)} - \frac{\sin(m+n)z}{2.(m+n)} + \text{constant}.$$

5°. $\int dz.\ \sin mz.\ \cos nz = \frac{1}{2}\int dz\,(\sin(m+n)z + \sin(m-n)z)$

$$= -\frac{\cos(m+n)z}{2.(m+n)} - \frac{\cos(m-n)z}{2(m-n)} + \text{constant}.$$

To integrate the differential expressions $\frac{dz}{\sin z}$; $\frac{dz}{\cos z}$; $\frac{dz.\cos z}{\sin z}$; $\frac{dz.\sin z}{\cos z}$, $\frac{dz}{\sin z.\cos z}$.

1°. $\frac{dz}{\sin z} = \frac{dz.\sin z}{1-\cos^2 z} = \frac{-d.\cos z}{1-\cos^2 z} = -\frac{1}{2}\left\{\frac{1}{1+\cos z} + \frac{1}{1-\cos z}\right\} d.\cos z,$

$$\therefore \int \frac{dz}{\sin z} = -\frac{1}{2}\log.(1+\cos z) + \frac{1}{2}\log.(1-\cos z) = \log.\sqrt{\frac{1-\cos z}{1+\cos z}}$$

$$= \log.\tan\tfrac{1}{2}z + \text{constant}.$$

2°. For z, using $\frac{\pi}{2} - z$, we have

$$\int \frac{dz}{\cos z} = \int \frac{-d.\left(\frac{\pi}{2}-z\right)}{\sin\left(\frac{\pi}{2}-z\right)} = \log.\sqrt{\frac{1+\sin z}{1-\sin z}} = \log.\tan\left(\frac{\pi}{4}+\frac{z}{2}\right) + \text{constant}.$$

3°. $\int \frac{dz.\cos z}{\sin z} = \log.\sin z + \text{constant.}$

4°. $\int \frac{dz.\sin z}{\cos z} = -\log.\cos z + \text{constant.}$

5°. Adding together the two last integrals, we have

$$\int \frac{dz}{\sin z.\cos z} = \log.\tan z + \text{constant.}$$

To find the value of the expression $\sin v.\int \Omega.\ \cos v.\,dv - \cos v.$ $\int \Omega \sin v.\,dv$, *where* $\Omega = \cos mv$.

$$\sin v \int \Omega.\cos v.\,dv = \sin v.\ \tfrac{1}{2}\int (\cos(m+1).\,v.\,dv + \cos(m-1)\,v.\,dv) =$$

$$\frac{\sin v.\sin(m-1)\,v}{2.(m+1)} + \frac{\sin v.\sin(m-1)\,v}{2.(m-1)} + C,$$

where $C=0$, if we take the integral cypher when $v=0$.

Again,

$$\cos v.\int \Omega \sin v.\,dv = \cos v.\ \tfrac{1}{2}\int (\sin(m+1)\,v - \sin(m-1)\,v\,dv) =$$

$$\frac{\cos v.\cos(m-1)\,v}{2.(m-1)} - \frac{\cos v.\cos(m+1)\,v}{2.(m+1)} + C'.\cos v,$$

and taking the integral from the same limit,

$$C' = \frac{1}{m^2-1},$$

$\therefore$ adding the two expressions, we find the value required

$$=\frac{-\cos mv}{2.(m-1)}+\frac{\cos mv}{2.(m+1)}+\frac{\cos v}{m^2-1}=\frac{\cos v}{m^2-1}-\frac{\cos mv}{m^2-1}.$$

Clairaut *Theorie de la lune*, page 9.

Required the value of $\int dz. \cos mz. \cos nz. \cos pz.$

$$\cos mz. \cos nz. \cos pz = \tfrac{1}{2}\{\cos(m-n)z+\cos(m+n)z\}. \cos pz =$$
$$\tfrac{1}{4}\{\cos(m-n-p)z+\cos(m-n+p)z\}+$$
$$\tfrac{1}{4}.\{\cos(m+n-p)z+\cos(m+n+p)z\}.$$

Hence the integral is obvious.

APPENDIX.

1. Any number, a, being assumed, and any other number, b, then b may be considered as a power of a:—*e. g.* if $a=2$, and $b=8$, then $a^3=b$. Retaining the number a, and adopting some third number, c, this may be considered as some power of a; for instance, if $a=2$, and $c=16$, then $a^4=c$; and in general any number, y, may be expressed as a power of a by the equation $y=a^x$.

2. In such an equation the index x is called the *logarithm* of y, and the fixed number a is called the *base* of the system. Hence the logarithm of a number may be defined, "the exponent of the power to which it is necessary to raise some fixed base, that it may be equal to the number."

3. The base a may have any value but unity. Napier, the inventor of logarithms, used as a base 2.71828182, &c. which was afterwards changed by Briggs for the base 10. The reasons and comparative advantages of both shall be explained.

4. The advantages derived from the invention of logarithms are many and important. The primary and more obvious benefits that they confer on science, by abridging numerical calculations, are suggested by the following properties:—

1°. *The logarithm of a product is the sum of the logarithms of its factors.*

Let $a^x = y$, $a^{x'} = y'$, $a^{x''} = y''$, &c. then

$$a^{x+x'+x'', \&c.} = y \cdot y' \cdot y'', \&c.$$

$\therefore$ by the definition of logarithms, $x+x'+x''$, &c. is the logarithm of $y \cdot y' \cdot y''$, &c.

2°. *The logarithm of a quote is the difference of the logarithms of the dividend and divisor.*

Let $a^x = y$, $a^{x'} = y'$, then

$$a^{x-x'} = y \div y',$$

$\therefore$ $x - x'$ is the logarithm of $y \div y'$.

3°. *The logarithm of an* m*th power is* m *times the logarithm of the root.*

Let $a^x = y$, then

$$a^{mx} = y^m,$$

$\therefore$ mx is the logarithm of y^m.

4°. *The logarithm of an* m*th root is the* m*th part of the logarithm of the power.*

Let $a^x = y$, then

$$a^{\frac{x}{m}} = y^{\frac{1}{m}}, \quad \therefore \&c.$$

Hence it may be seen how, if knowing the numbers, we knew their logarithms, and *vice versâ*; complex multiplications and divisions might be performed by additions and subtractions; and involutions and evolutions by multiplications and divisions.

5. Logarithms are defined by Briggs, "*The equidifferent companions of proportional numbers.*"

Let $a^x = y$, $a^{x'} = y'$, $a^{x''} = y''$; then if $y : y' :: y' : y''$, we have

$$x + x'' = 2x',$$

$\therefore x, x', x''$, are equidifferent. They were called logarithms by Napier, because being taken equidifferent they exhibit numbers in the same ratio. *Arithmetica logarithmica ab Adriano Vlacq*. p. 1.

6. The integral part of a logarithm shews the order of units the number of which it is the logarithm ranks with. *E. g.*—Supposing the base 10, if the integral part be 0, 1, 2, &c. the number is accordingly between 1 and 10, between 10 and 100, between 100 1000, &c. respectively. The integral part part is called the *characteristic*, and has in it as many units as there are digits in the number *minus* one.

7. The logarithm of a number, and the logarithm of the product of that number by the m^{th} power of the base, have the same decimal part.

Let $y = a^{\phi \cdot \phi'}$, then

$$y \cdot a^m = a^{m + \phi \cdot \phi'},$$

$\therefore$ they differ only in the characteristic.

8. The logarithms of proper fractions are negative, being the difference of the logarithms of the numerator and denominator. To find the fraction to which a given negative logarithm corresponds:—"Subtract the negative logarithm, without any regard to its sign, from the log. of 10^m, viz. m; find the number that corresponds to this difference, and in it move the decimal point m places to the left."

Let $\frac{m}{n}$ be the fraction, then

$$\frac{m}{n} = \frac{10^m}{\left(\frac{n}{m}\right)} \cdot \frac{1}{10^m}$$

but

$$\log. \frac{10^m}{\left(\frac{n}{m}\right)} = m - l. \frac{n}{m},$$

and ∴ *vice versâ* the number to which this latter log. corresponds is $\frac{10^m}{\left(\frac{n}{m}\right)}$, which, to give the true fraction $\frac{m}{n}$, must be divided by 10^m.

9. A fractional logarithm may be written wholly negative, or wholly positive, or partly negative and partly positive.

Wholly negative, if considered as the difference of the logarithms of the numerator and denominator.

If to the decimal of this there be added unity, and unity subtracted from the characteristic, it will be negative as to the characteristic, and positive as to the decimal.

And this is rendered wholly positive, by writing for the characteristic what it wants of 10, and compensating for this by a subsequent subtraction of 10.

As an example of all three methods, the logarithm of

$$\tfrac{75}{100} = 1.87506 - 2.00000 = -0.12494.$$

The objection to this method is, that the number found in the tables to answer such a logarithm must be considered the denominator of a fraction whose numerator is unity, and thus we are encumbered with fractions, *not* decimal fractions.

In the second method,

$$\log. \tfrac{75}{100} = -1 + 1 - 0.12496 = \bar{1}.87506.$$

The objection to this method is, that the negative sign is still

retained, and there is danger of confusion in the contrary operations of addition of decimals and subtraction of characteristics.

In the third method,

$$\log. \tfrac{75}{100} = 9.87506.$$

This is the method most usually adopted, and the safety of it consists in this, that the result it gives is 10^{10} times too great, and $\therefore$ cannot possibly but be allowed for. In the addition then of the logarithms of decimal fractions, the tens must be all rejected, as many as there are decimals, if it can be done without making the result negative, if not, one must be retained, and this one of course attended to in the final result.

10. There exists between logarithms of different systems an invariable ratio.

Let $a^x = y = A^X$, then

$$x . la = X,$$

$\therefore x = \frac{1}{la} . X$, the same for every value of y. The quantity $\frac{1}{la}$ is called the *modulus* of the new system relative to the old.

11. Nothing has as yet been said on the solution of the equation $y = a^x$. If $x = 0$, $y = 1$, and for values of x above cypher y increases; when x becomes negative, y becomes fractional, and as y becomes small, x increases negatively *ad inf.* Hence when y becomes 0, x is infinite, though negative. From this it appears, that it is impossible to expand x in a series ascending by the *positive* powers of y, for the supposition of $y = 0$ would destroy the series, whereas x then becomes infinite. Such is not the case, however, with log. $(1+y)$, for when $y = 0$, $x = \log. 1 = 0$.

Assume then

$$l(1+y) = My + Ny^2 + Py^3 + \&c. \qquad (A)$$

$$l(1+y)^2 = 2 . l(1+y) = 2\,My + 2\,Ny^2 + 2\,Py^3 + \&c.$$

But by the general form (A), we have

$$l.(1+2y+y^2)=M.(2y+y^2)+N(2y+y^2)^2+\&c.$$
$$=2My+(M+4N)y^2+(4N+8P)y^3+\&c.$$

Equating the co-efficients of like powers of y in the two expansions of $l(1+y)^2$, we have

$$M=M;\ N=-\frac{M}{2};\ P=\frac{M}{3};\ Q=-\frac{M}{4};\ \&c.$$

∴ we have

$$l.(1+y)=M.\{y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\&c.\} \qquad (a)$$

In this series the value of M is still indeterminate; let $y=(a-1)$, then we have

$$l.(1+a-1)=1=M.\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.\} \qquad (b)$$

From this series we can obtain M, and thus the developement of $l(1+y)$ becomes completely determined.

The quantity M has an immediate relation to a, so that different systems of logarithms may be formed by arbitrarily adopting either M or a, and determining the other, by equation (b) or some such.

If $M=1$, the system will be that adopted by Napier. The logarithms of this system are commonly called *natural* or *hyperbolic* logarithms: for the hyperbolic areas between the Asymptotes, which in general are represented by logarithms in any system, belong in this case to the most remarkable and simplest of hyperbolæ, viz. the equilateral.

12. If $y <$ or $=1$, series (a) is convergent, but otherwise it becomes divergent, and ∴ unfit for the computation of logarithms. To remedy this inconvenience we must have recourse to transformations.

If $y<0$, the series (a) becomes

$$l(1-y)=M.\left\{-y-\tfrac{1}{2}y^2-\tfrac{1}{3}y^3-\&c.\right\} \qquad (c)$$

From series (c) and (a), by subtraction, we have

$$l\,\frac{1+y}{1-y}=2\,M.\left\{y+\tfrac{1}{3}y^3+\tfrac{1}{5}y^5+\&c.\right\} \qquad (d)$$

From this, using z for $\frac{1+y}{1-y}$, we have

$$lz=2\,M.\left\{\left(\frac{z-1}{z+1}\right)+\tfrac{1}{3}\left(\frac{z-1}{z+1}\right)^3+\tfrac{1}{5}\left(\frac{z-1}{z+1}\right)^5+\&c.\right\} \qquad (e)$$

13. If in series (c) we make $y=1$, we have

$$l(o)=-\infty,$$

which shews that negative numbers can have no logarithms, for all the positive numbers from o to ∞ engage as logarithms all the real numbers from $-\infty$ to $+\infty$.

14. If in series (d) we make $\frac{1+y}{1-y}=a$, we have

$$\frac{1}{M}=2.\left\{\left(\frac{a-1}{a+1}\right)+\tfrac{1}{3}\cdot\left(\frac{a-1}{a+1}\right)^3+\&c.\right\} \qquad (f)$$

This is a more convergent series for finding M than series (b).

15. Knowing log. n, if we wish to find that of $(n+dn)$; dn being a small change on n, we can obtain from (d) a very convergent series for the purpose.

Let $\frac{n+dn}{n}$ be written for $\frac{1+y}{1-y}$, then

$$y=\frac{dn}{2n+dn},$$

∴ we have

$$l\frac{n+dn}{n}=l.(n+dn)-ln=2M.\left\{\frac{dn}{2n+dn}+\tfrac{1}{3}.\left(\frac{dn}{2n+dn}\right)^3+\right\} \quad (g)$$

This very convergent series may be adapted to numbers $< n$, by taking dn, negative. Cagnoli claims this series as his own, and gives himself no small degree of credit for the discovery, though it is but another form of the series (d) or (e), which were known long before, as they were deduced by Newton and Gregory from the hyperbola, and afterwards by Halley from the composition of ratios.

The series (g) may be exemplified in finding the logarithms of 101 and 99. If $a = 10$, the value of M from series (f) is

0.43429448190325 &c.

and ∴ from series (g)

$$\log.(101)=2+2M.\left\{\tfrac{1}{201}+\tfrac{1}{3}(\tfrac{1}{201})^3+\&c.\right\}$$

$$\log.99=2-2M.\left\{\tfrac{1}{199}+\tfrac{1}{3}(\tfrac{1}{199})^3+\&c.\right\}$$

16. If for y in series (a) we use $(\sqrt[m]{y}-1)$, we have

$$l\sqrt[m]{y}=\frac{1}{m}ly=M.\left\{(\sqrt[m]{y}-1)-\tfrac{1}{2}(\sqrt[m]{y}-1)^2+\&c.\right\} \quad (h)$$

This series can be made convergent *sine limite*, by taking m sufficiently large, for thus $\sqrt[m]{y}$ can be made to differ from unity by a quantity < any assignable. We should select m some number in the series 2, 4, 8......2^n, that we may have only repeated extractions of square roots to perform, and we can always so contrive it, that the first term of the series should give a sufficiently near value. Briggs stumbled on this method, but has explained in no satisfactory way whence its accuracy proceeded. As an instance of the method, we shall take one afforded by Briggs himself. He found it particularly necessary, for a reason to be mentioned hereafter, to be able to compute the Napierian logarithm of 10. For this purpose he extracted, fifty-four times in order, the square root of 10, and thus obtained a value for $(\sqrt[m]{y}-1)$, whose first fifteen decimals were cyphers. Hence in $(\sqrt[m]{y}-1)^2$ the first 30

or 31 decimals were *necessarily* cyphers; and ∴ should even m have 20 integer digits in its composition, the quantity $\frac{m}{2}({}^{m}\sqrt{y}-1)^2$ would still have the eleven first decimals cyphers, and ∴ the eleven first decimals of the value of ly, obtained from the first term of the series, uninjured by the rejection of $({}^{m}\sqrt{y}-1)^2$. And indeed 2^{54} has only 17 digits in its composition.

Series (h) is due to Lagrange.—Vide *Calc. des Fonct.* page 33. It would be worth while to compare this beautiful series, and Lagrange's manner of arriving at it, with the clumsy account given by Briggs of the first term of it in the sixth chapter of the *Logarithmica Arithmetica.* It would be difficult to find a more striking instance of the superiority of modern analysis.

17. Lagrange has moreover deduced another, in which all the terms are positive, and that simply by changing the sign of m, for

$$({}^{-m}\sqrt{y}-1)=-\left(1-\frac{1}{{}^{m}\sqrt{y}}\right);$$

by introducing this in series (c), and observing that $ly=-m.l.{}^{-m}\sqrt{y}$, we find

$$ly=m.\,M.\left\{\left(1-\frac{1}{{}^{m}\sqrt{y}}\right)+\tfrac{1}{2}.\left(1-\frac{1}{{}^{m}\sqrt{y}}\right)^2+\tfrac{1}{3}.\left(1-\frac{1}{{}^{m}\sqrt{y}}\right)^3+\right\} \quad (i)$$

All the terms in this being additive,

$$ly>m\,M.\left(1-\frac{1}{{}^{m}\sqrt{y}}\right);$$

but in series (h), since the second term is greater than the third, the second and third combined give a negative result; so do also the fourth and fifth, &c., ∴ $ly<m\,M.({}^{m}\sqrt{y}-1)$.

18. Hence we have two limits to the value of ly; take the difference of those limits and it comes out

$$m\,M.\frac{({}^{m}\sqrt{y}-1)^2}{{}^{m}\sqrt{y}}$$

This difference, by taking m sufficiently large, can be made $<$ any assignable quantity; and $\therefore$ *a fortiori*, the value of ly, obtained from the first term of series (h) or (i), differing from the true value by a quantity $<$ any assignable.

19. Any or all of the preceding series give the logs. for Napier's system, by merely supposing $M=1$.

20. If $y=10$ and $M=1$, then ly is the log. of Briggs' base in Napier's system. Denoting Briggs' base by a, and marking with a trait the logs. of Napier's system, we have

$$l' a = m.(\sqrt[m]{a}-1),$$

$\therefore$ we have

$$M. l' a = 1$$

21. It shall be shewn further on, that M (the co-efficient of Napier's logs. to give those of any other system) is the log. of Napier's base in the new system.

Let e denote Napier's base, then in the systems of Briggs and Napier, we have

$$le. l' a = 1.$$

22. In series (d), if $y=1$, one side of the equation becomes infinite, and the other the series $1+\frac{1}{2}+\frac{1}{3}+$ &c. *ad inf.* $\therefore$ the sum of this series is infinite.

23. By series (c) we have

$$l'(1-y) = -\left\{y+\frac{y^2}{2}+\frac{y^3}{3}+\text{\&c.}\right\}$$

in which, using $\frac{1}{z}$ for $(1-y)$, we have

$$l'\left(\frac{1}{z}\right) = -\left\{\frac{(z-1)}{z}+\frac{1}{2}\left(\frac{z-1}{z}\right)^2+\frac{1}{3}\cdot\left(\frac{z-1}{z}\right)^3+\text{\&c.}\right\} \qquad (j)$$

Should z become infinite in this series, $l'\frac{1}{z}=-l'z$ is also infinite; and the series becomes,

$$-\{1+\tfrac{1}{2}+\tfrac{1}{3}+\&c.\}$$

$\therefore$ the sum of this series, commonly called the *harmonic series*, is infinite.*

A very important consequence may be deduced from this, viz. that every series, the terms of which decrease more rapidly than those of the harmonic series, must necessarily have a limit; for $l'\left(\frac{1}{z}\right)$ being a finite quantity as long as z is not infinite, it is necessary that the series of fractions $\frac{z-1}{z}+\frac{1}{2}\cdot\left(\frac{z-1}{z}\right)^2+\&c.$ which forms a convergent series must have a limit, however near to unity be the fraction $\frac{z-1}{z}$.

24. Exceedingly convergent series for the computation of lo-

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* The following method of proving that the harmonic series is infinite, has been suggested by a friend.

Let $S=1+\frac{1}{2}+\frac{1}{3}$ &c. dividing by 2, we have

$$\frac{S}{2}=\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\&c.$$

subtracting this from the former,

$$\frac{S}{2}=1+\tfrac{1}{3}+\tfrac{1}{5}+\&c.$$

again, subtracting these values of $\frac{S}{2}$, we find

$$0=\frac{1}{1.2}+\frac{1}{3.4}+\&c.$$

an absurdity that can only be reconciled by supposing S infinite.

garithms of the successive numbers, may be formed from those here given. Some of these we shall accordingly proceed to introduce.

In series (e) let $z=\frac{x^2}{x^2-1}$, then

$$lz=2\,lx-l(x-1)-l(x+1),$$

$$\therefore l.(x+1)=2\,lx-l(x-1)-2\,M.\left\{\frac{1}{2\,x^2+1}+\tfrac{1}{3}.\frac{1}{(2\,x^2+1)^3}+\quad\right\} (1)$$

This formula is very convergent, for if $x=1000$, it proceeds according to the odd powers of $\frac{1}{1999999}$.

In the same series, let $z=\frac{x^3-3\,x+2}{x^3-3\,x-2}=\frac{(x-1)^2.\,(x+2)}{(x+1)^2.\,(x-2)}$, then

$$\left.\begin{array}{l} l(x+2)+2\,l(x-1)-2\,l(x+1)-l(x-2)=2\,M. \\ \quad\left\{\frac{2}{x^3-3\,x}+\tfrac{1}{3}.\left(\frac{2}{x^3-3\,x}\right)^3+\&c.\right\} \end{array}\right\} (2)$$

This formula, due to Borda, was one of the first of its kind. Others have since been formed that converge still faster; as for instance, that of M. Haros, in which z in formula (e) is replaced by

$$\frac{x^4-25\,x^2}{x^4-25.\,x^2+144},$$

and $\therefore$

$$\frac{z-1}{z+1}=\frac{72}{x^4-25\,x^2+72}.$$

The factors of the numerator of z are x^2, $x-5$, $x+5$, and of the denominator $x-3$, $x+3$, $x-4$, $x+4$; whence we can find the logarithm of $x+5$, by the logarithms of six preceding numbers, and the series

$$2\,M.\left\{\frac{72}{x^4-25\,x^2+72}+\&c.\right\}$$

It is obvious that series still more convergent may be formed by taking fractional functions of an higher degree, to substitute for z, and the choice of such is only subject to the following conditions:—*To find two equations which, not differing except in their last term, should have commensurable and integer roots.*

25. Amongst the many series for logarithms that have been given, we have not as yet met with one that proceeds according to the powers of the number. It is evident that such a series cannot be found proceeding exclusively by the positive powers of the number, for when the number vanishes, the series would be restricted to its first term which should ∴ be infinite; nor by the *negative*, for then the same thing would happen where the number became infinite. We may, however, obtain a series that will proceed at the same time by both the positive and negative powers.

$$l.\,(1+u)=M\left\{u-\frac{u^2}{2}+\frac{u^3}{3}-\&c.\right\}$$

$$l.\left(1+\frac{1}{u}\right)=M\left\{u^{-1}-\frac{u^{-2}}{2}+\frac{u^{-3}}{3}-\&c.\right\}$$

Subtracting these series, the difference of the left hand members is lu, ∴

$$lu=M.\,\{(u-u^{-1})-\tfrac{1}{2}.\,(u^2-u^{-2})+\tfrac{1}{3}\,(u^3-u^{-3})-\&c.\} \qquad (k)$$

This remarkable series is due to Lagrange. We have had some remarkable consequences from it in Chap. II. Part II. relative to circular functions.

26. An approximate value for the logarithm of a number may be obtained as follows in a continued fraction.

Let $a^x=y$, then x can easily be seen to be some number between n and $n+1$; let then $x=n+\frac{1}{x'}$; substituting this value in the equation, we find

$$a^{n+\frac{1}{x'}}=y,$$

whence

$$\left(\frac{y}{a^n}\right)^{x'} = a,$$

or

$$c^{x'} = a.$$

Operating on this equation, as on the proposed, we shall find that x' is comprised between n' and $n'+1$, whence

$$x' = n' + \frac{1}{x''}.$$

By continuing this operation we shall find

$$x = n + \cfrac{1}{n' + \cfrac{1}{n'' + \cfrac{1}{n''' + \&c.}}}$$

As an example of this method, let $2^x = 6$, then $x > 2 < 3$, putting $x = 2 + \frac{1}{x'}$, we find, using it for x in the given equation,

$$\left(\frac{3}{2}\right)^{x'} = 2.$$

Hence

$$x' > 1 < 2;$$

let then $x' = 1 + \frac{1}{x''}$, and we obtain from the second equation

$$\left(\frac{4}{3}\right)^{x''} = \frac{3}{2}.$$

Hence

$$x'' > 1 < 2;$$

let then $x'' = 1 + \frac{1}{x'''}$, and there results from the third equation

$$\left(\frac{9}{8}\right)^{x'''} = \frac{4}{3}.$$

Hence

$$x''' > 2 < 3 \text{ \&c.}$$

Collecting the results obtained, we have

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{x^{\mathrm{IV}}}}}}$$

27. Applying the ordinary process of converting a series into a continued fraction, any one of the series that have been given may be exhibited in this form. For instance, series (*d*) so treated gives us

$$\tfrac{1}{2} l. \frac{1+x}{1-x} = \cfrac{1}{\cfrac{1}{x} - \cfrac{1}{\cfrac{3}{x} - \cfrac{4}{\cfrac{5}{x} - \cfrac{9}{\cfrac{7}{x} - \cfrac{16}{\cfrac{9}{x} - \cfrac{25}{\cfrac{11}{x} - \text{\&c.}}}}}}}$$

28. We have as yet said nothing of the developement of the number in terms of the logarithm, or in other words, of expressing a^x in a series ascending by the powers of x. For this let

$$a^x = h + kx + lx^2 + mx^3 + nx^4 + \text{\&c.} \qquad (\epsilon)$$

Squaring both sides of this equation, and arranging according to the powers of x,

$$a^{2x}=h^2+2hkx.+\left.\begin{matrix}k^2\\+2hl.\end{matrix}\right\}x^2+\left.\begin{matrix}2hm\\+2kl\end{matrix}\right\}x^3+\left.\begin{matrix}l^2\\+2km\\+2hn\end{matrix}\right\}x^4+\&c.$$

And from series (l), by using $2x$ for x

$$a^{2x}=h+2kx+4lx^2+8mx^3+16nx^4+\&c.$$

These two developements of a^{2x} must be identical, ∴ the co-efficients of like powers of x must be equal. Hence we obtain a series of equations,

$$h^2=h,\ 2hk=2k,\ k^2+2hl=4l,\ 2hm+2kl=8m,\ \&c.$$

from which

$$h=1,\ k=k,\ l=\frac{k^2}{1.2},\ m=\frac{k^3}{1.2.3},\ \&c.$$

∴ we have

$$a^x=1+\frac{kx}{1}+\frac{k^2x^2}{1.2}+\frac{k^3x^3}{1.2.3}+\&c. \qquad (l)$$

The co-efficient k remains still to be determined. Using for a, $(1+b)$, we have

$$(1+b)^x=1+\frac{x}{1}.b+\frac{x.(x-1)}{1.2}.b^2+\frac{x.(x-1).(x-2)}{1.2.3}.b^3+\&c. \qquad (l')$$

which, if arranged according to the powers of x, would give as the co-efficient of its first power,

$$b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\&c.=(a-1)-\frac{(a-1)^2}{2}+\&c.=k$$

of series (l). But

$$(a-1)-\frac{(a-1)^2}{2}+\&c.=\frac{1}{M}$$

by series (b), $\therefore$

$$k=\frac{1}{M}.$$

Thus by series (l) we have the law of the co-efficients, and by ($\prime$) the value of the first co-efficient.

If in series (l) we use M for x, we shall have

$$a^M=1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.$$

Supposing then $M=1$, we have

$$a=e$$

(then the base of Napier's system), $\therefore$

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\&c.=2.7182818$$

the result obtained by adding the first ten terms.

Since $a^M=e$, we have, *in any system*,

$$M=\frac{le}{la};$$

in the system of Briggs,

$$M=le;$$

in the system of Napier,

$$M=\frac{1}{la'};$$

$\therefore$ we have three forms for a^x, viz.

$$a^x = 1 + \left(\frac{la}{le}\right) \cdot \frac{x}{1} + \left(\frac{la}{le}\right)^2 \cdot \frac{x^2}{1.2} + \&c. \qquad (m)$$

$$a^x = 1 + (l'a) . \frac{x}{1} + (l'a)^2 . \frac{x^2}{1.2} + \&c. \qquad (n)$$

$$a^x = 1 + \left(\frac{x}{le}\right) + \left(\frac{x}{le}\right)^2 \cdot \frac{1}{1.2} + \&c. \qquad (o)$$

29. In the system of Napier, series (l) becomes

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c. \qquad (p)$$

If x be negative,

$$e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} \qquad (q)$$

Adding these two series, we have

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \&c.$$

Subtracting, we have

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.$$

We have already, in pages 105 and 135, seen the valuable and important consequences deducible from these forms.

30. Since $xla = la^x$, series (m) may be written

$$a^x = 1 + \left(\frac{la^x}{le}\right) + \left(\frac{la^x}{le}\right)^2 \cdot \frac{1}{1.2} + \&c. \qquad (r)$$

and series (n) and (o) may of course be similarly transformed. Hence from series (n), by using n for a^x, we have

$$n = 1 + l' n + \tfrac{1}{2}.\,(l' n)^2 + \frac{1}{2.\,3}.\,(l' n)^3 + \&c. \qquad (s)$$

Changing n in this series into x^{nx}, we find

$$x^{nx} = 1 + nx\,(l' x) + \frac{n^2\,x^2}{2}.\,(l' x)^2 + \&c. \qquad (t)$$

From series (s) we have again

$$x^m = 1 + m.\,l' x + \tfrac{1}{2}\,m^2.\,(l' x)^2 + \&c.$$

$$r^m = 1 + m.\,l' r + \tfrac{1}{2}\,m^2.\,(l' r)^2 + \&c.$$

whence we have the following curious developement

$$x^m - r^m = \frac{m.\,(l' x - l' r)}{1} + \frac{m^2.\,(l' x^2 - l' r^2)}{2} + \&c. \qquad (u)$$

31. It may not be inappropriate to introduce here the series that have been given by Legendre for the solution of the third case of plane triangles.

We have

$$\frac{\sin B}{\sin (B+C)} = \frac{b}{a}, \quad \therefore \frac{\sin B}{\cos B} = \frac{b.\,\sin C}{a - b.\,\cos C}.$$

Introducing for the sines and cosines their imaginary values given in page 105, we have

$$\frac{e^{B\sqrt{-1}} - e^{-B\sqrt{-1}}}{e^{B\sqrt{-1}} + e^{-B\sqrt{-1}}} = \frac{b.\,(e^{C\sqrt{-1}} - e^{-C\sqrt{-1}})}{2\,a - b.\,(e^{C\sqrt{-1}} + e^{-C\sqrt{-1}}}$$

whence

$$e^{2B\sqrt{-1}} = \frac{a - b.\,e^{-C\sqrt{-1}}}{a - b.\,e^{C\sqrt{-1}}}.$$

Taking the logarithms of each member, and expanding the two parts of the second member by formula (c), we shall have

$$2\,B\sqrt{-1}=\frac{b}{a}\,(e^{C\sqrt{-1}}-e^{-C\sqrt{-1}})+\frac{b^2}{2a^2}.\,(e^{2C\sqrt{-1}}-e^{-2C\sqrt{-1}})+$$

or,

$$B=\frac{b}{a}.\ \sin C+\frac{b^2}{2a^2}.\ \sin 2\,C+\frac{b^3}{3a^3}.\ \sin 3\,C+\&c.$$

A similar series holds for A.

With respect to the third side c, we have

$$c^2=a^2-2\,ab\cos C+b^2=(a-b.\,e^{C\sqrt{-1}})\,(a-b.\,e^{-C\sqrt{-1}}).$$

Taking the logarithms of each member,

$$2\,lc=la-\frac{b}{a}.\ e^{C\sqrt{-1}}-\frac{b^2}{2a^2}.\ e^{2C\sqrt{-1}}-\frac{b^3}{3a^3}.\ e^{3C\sqrt{-1}}-\&c.$$

$$+la-\frac{b}{a}.\ e^{-C\sqrt{-1}}-\frac{b^2}{2a^2}.\ e^{-2C\sqrt{-1}}-\frac{b^3}{3a^3}.\ e^{-3C\sqrt{-1}}-\&c.$$

whence we shall have

$$lc=la-\frac{b}{a}.\ \cos C-\frac{b^2}{2a^2}.\cos 2\,C-\frac{b^3}{3a^3}.\ \cos 3\,C-\&c.$$

32. With respect to the systems of Briggs and Napier, it may be remarked that the latter (like circular functions to radius unity) has the advantage as long as we have to do with mere theory, and the use of symbolical language. When we descend to actual computation and the deduction of practical results, the system of Briggs is superior. The chief advantage is, that $l.\ N$ being known, we have that of $l\,10^m\times N$ by simply adding m; and such numbers as are formed by multiplying N by powers of 10 are obvious on inspection, hence the logs. of all such may be omitted in the tables. It is true that $l.\ 3^m\times N$, supposing 3 the base, is also found from that of N, by adding m, but such numbers are not obvious

on inspection; the logs. ∴ of such numbers cannot be safely omitted in the tables. This circumstance arises from the fact that the *radix* of our numeral scale is 10: Woodhouse seems to be the first that noticed this advantage. We shall now proceed to give some short account of the tables.

In the tables of Vlacq the logarithms of all numbers up to 500000 are set down in their order to seven places of decimals, and there is a column of differences by which we can find the logarithm of any number between n and $n+1$; suppose $n.n'$ by the proportion $1 : .n :: d : d \times .n$, which gives the difference that corresponds to the decimal $.n$. These tables, however, from their immense size, are inconvenient for ordinary use.

33. In the tables of Hutton and Sherwin the logarithms of numbers consisting of five places are actually set down, and there is a column of proportional parts near the margin of each page for finding the logarithms of higher numbers. We shall shew how such parts may be found.

Let the number be N, and n the number formed of the five first digits, then

$$N = 10^x . n + y,$$

where y is a number consisting of x digits,

$$l.\ N = l.(10^x . n + y) = l.\ 10^x . n + l. \left(1 + \frac{y}{10^x . n}\right)$$

$$= l.\ 10^x . n + M. \frac{y}{10^x . n} q.\ p.$$

∴ the correction

$$= M. \frac{y}{10^x . n};$$

but

$$\frac{M}{n} = l.\ (n+1) - ln$$

∴ the correction

$$=\frac{y}{10^x}(l.\,(n+1)-l.\,n).$$

34. It is also necessary to have a rule for finding the number corresponding to a logarithm, not exactly to be found in the tables.

Let L be the proposed log., l, l', (one greater, and the other less than L), the logs. of the n and $n+1$. Let N be the number sought, then

$$l.\,N=l.\,(n+x)=l+M.\frac{x}{n}, \text{ nearly},$$

$$\therefore x=\frac{n.\,(L-l)}{M}.$$

Again,

$$l.\,(n+1)=l+\frac{M}{n};$$

$$\therefore l'-l=\frac{M}{n}; \quad \therefore x=\frac{L-l}{l'-l};$$

and $\therefore N$ is found.

We have thus given probably a sufficient detail of the theory of logarithms; as for the practice of them the reader will find it necessary to refer to the tables and the instructions prefixed thereto.

35. We shall next proceed to give some account of the mode of computing the numerical ratio of the perimeter of a circle to its diameter, and of finding the actual values of Trigonometrical lines both natural and logarithmic.

We have, page 107, arrived at the series for A in terms of $\sin A$, viz.

$$\sin A+\frac{\sin^3 A}{1.\,2.\,3}+\frac{3.\,\sin^5 A}{2.\,4.\,5}+\ldots\ldots\ldots\frac{3.\,5\ldots\ldots(2\,n-1).\,\sin A^{2\,n+1}}{2.\,4.\,6\ldots2\,n\,(2\,n+1)}$$

in which, if we put $A=30°$, and multiply the result obtained by 6, we may obtain the semi circumference to radius unity with any degree of exactness.

This computation has actually been made to 127 places of decimals. However, as a sufficiently near value, we may take $180°=R\,(3{,}141592)$.

Knowing the numerical value of 180° in parts of the radius, we of course can find the values of all other arcs.

The series for A in terms of tan A may also be used to compute the values of arcs of a circle, but the convergence is much too slow unless the tangent be very small. In order to find series for small tangents, from which to find $\frac{\pi}{4}$, let A be an arc of a known small tangent; let $\frac{\pi}{4}=m.A-z$, then z is the arc, the expression for which in terms of its tangent, is to be subtracted from the expression for mA in terms of the tangent of A. To find tan z, if we know tan mA, we have

$$\tan z=\frac{\tan mA-1}{\tan mA+1};$$

but tan mA can be found by knowing tan A.

Let, for instance, tan $A=\frac{1}{5}$, and $m=4$, then

$$\tan 2A = \tfrac{5}{12}, \text{ and } \tan 4A = \tfrac{120}{119},$$

$\therefore \tan z=\frac{1}{239}$, $\therefore \frac{\pi}{4}=4$ times the arc whose tangent $=\frac{1}{5}$—arc whose tangent $=\frac{1}{239}$. This is Machin's formula.

Hence,

$$\frac{\pi}{4}=\left\{\begin{array}{l} 4.\left\{\frac{1}{5}-\frac{1}{3}.\frac{1}{5^3}+\frac{1}{5}.\frac{1}{5^5}-\frac{1}{7}.\frac{1}{5^7}+\&c.\right. \\ -\left\{\frac{1}{239}-\frac{1}{3}.\frac{1}{239^3}+\&c.\right. \end{array}\right\}$$

If $m=1$, and tan $A=\frac{1}{2}$, then

$$\tan z = -\tfrac{1}{3}, \therefore \frac{\pi}{4} = A + z,$$

or by the series

$$\frac{\pi}{4} = \left\{ \begin{array}{l} \frac{1}{2} - \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{5} \cdot \frac{1}{2^5} - \frac{1}{7} \cdot \frac{1}{2^7} + \&c. \\ \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} - \frac{1}{7} \cdot \frac{1}{3^7} + \&c. \end{array} \right\}$$

This is the expression of Euler, *Vide Anal. inf.* p. 107' Vol. I.

Machin's was published before Euler's, and is far more convergent. Though Machin was remarkable for having computed to 100 places of decimals the circumference of a circle which Leudolph Van Ceulen had computed to 35, still his formula lay for a long time unnoticed.

The series for the arc in terms of the sine, as well as those of the sine and cosine in terms of the arc, were first given by Newton in his *analysis per Eq. Num. term inf. Vide* page 22, Editio Castilionei, Vol. I.; that for the arc in terms of the tangent is due to Gregory.

Newton's modes of deducing the above series are well worth attending to, as containing the germ of a very important point in analysis, viz. the rectification of curves, and conversely finding the lines by means of which this rectification is performed. He entitles the former, *inventio logitudinum curvarum*, and the latter, *inventio basis ex datâ longitudine curvæ.*

Newton gives a series for the arc of a quadrant, the chord being supposed unity, viz. $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} -$ &c. the computation of a quadrant by this series would be, he says, the work of 1000 years to 20 places of decimals, for it would require 5000000000 terms of the series to be computed. He says it would be still worse to compute by Gregory's series, and ∴ recommends to compute by his own series in terms of the sine. He afterwards, however, recommends to compute the arc 30°, from Gregory's series. —*Vide Newt. Opuscula*, page 345, tom. I.

Euler has shewn how numberless formulæ similar to those already given for computing by the tangent may be found, *Vide Nov. Com. Acad. Petrop.* tom ix. In Peacock's excellent collection of examples on the calculus, it is shewn how this problem depends upon assuming $a.\tan^{-1}.\frac{1}{x}+a'\tan^{-1}\frac{1}{x'}+a''\tan^{-1}\frac{1}{x''}$, &c. $=0$, and determining a, a', a'', &c. so that x, x', x''. &c. be integers.

36. The computation of Trigonometrical lines in parts of the radius may be effected to a great extent, by the simple formulæ that have been given in Chap. II. Part I. of this work.

From the formula $\cos A=\sqrt{\tfrac{1}{2}(1+\cos 2A)}$, we have (beginning with $2A=60°$, whose cosine is $\frac{1}{2}$), the cosines of the series of arcs $\frac{60}{2}$, $\frac{60}{4}$, $\frac{60}{8}$, &c. Thus we can continue to an arc the sub-multiple of 60° next to 1′, and find its cosine, and ∴ its sine, and thence on the principle that small arcs are as their sines, we may compute sin 1′.

Other formulæ might be used, viz.

$$\sin A=\tfrac{1}{2}\sqrt{1+\sin 2A}-\tfrac{1}{2}\sqrt{1-\sin 2A},$$

in which we can begin with using for $2A$, any arc of a known sine, for instance 30°, whose sine $=\frac{1}{2}$, or 60 whose sine $=\frac{\sqrt{3}}{2}$.

By the formulæ for sines, cosines, &c. of double arcs, triple arcs, sums and differences of arcs, this calculation can be promoted so as to include almost every arc in the quadrant.

37. Geometry may be called in to our aid, for instance, the side of a regular decagon in a circle, or 2. sin 18°, is the greater segment of the radius cut in extreme and mean ratio, ∴

$$\sin 18°=\frac{\sqrt{5}-1}{4},$$

and thus we have the series of all arcs formed by dividing 18° by the successive powers of 2.

In like manner the square of the side of a regular pentagon in a circle = sum of squares of sides of the regular hexagon and decagon in the same. This may be better used as a Theorem of verification, to try whether the sines of 18° and 36° have been rightly computed.

38. A very elegant set of formulæ of verification may be deduced from a Theorem which the reader will find geometrically and analytically demonstrated in the first chapter of the second part of this work, *vide* page 83. In the case of the pentagon using the halves of the chords, and expressing the Theorem in Trigonometrical language, we have

$$\sin(36^\circ+A)+\sin(72^\circ-A)=\sin(72^\circ+A)+\sin(36^\circ-A)+\sin A.$$

This is the formula given by Euler, *Anal. Inf.* but which he obtains in a very different manner. Legendre gives precisely the same, using for A, $90-A$; his $\therefore$ is

$$\sin(90^\circ-A)+\sin(18^\circ+A)+\sin(18^\circ-A)=\sin(54^\circ+A)+(54^\circ-A)$$

and Woodhouse verifies it by substituting numerical values for the cos of 36° and 72°. The reader can form as many others as he pleases from the Theorem above alluded to.

Particular values of A must be used, and thus we can try the accuracy of any set of tables by trying whether the values of the sine they give be such as to satisfy the formula.

As we can ascertain the value of any arc in parts of the radius, we can always find the values of sines, cosines, tangents, &c. from the series that have been given in Chap. II. Part II.

The reader will find no difficulty in demonstrating the following formula which has been given by M. Delambre:

$$\sin(A+1^\circ)=2\sin A-\sin(A-1^\circ)-4.\sin^2 30'.\sin A.$$

Knowing the sines of arcs up to A, this is an exceedingly convenient formula for continuing the operation, as we have seen already how we might compute the sine of 30′.

The best mode of computing the numerical values of tangents is, to compute the sines and cosines, and divide the former by the latter.

Tables constructed with the numerical results obtained in the preceding manner, are called *Trigonometrical tables in natural numbers.*

39. We shall next proceed to shew how logarithmic values may be obtained. From the values in natural numbers, we may find the logarithmic values, if we seek those values only to seven places of decimals. We can, however, easily compute them by independant methods. By the differential calculus we have

$$d.\,l.\sin A = dA.\cot A = d\,A.\left\{\frac{1}{A} - \frac{A}{3} - \frac{A^3}{3^2.\,5} - \&c.\right\}.$$

by series (4), page109. Integrating, and using the modulus M, we have

$$\log.\sin A = \log.\,A - M.\left\{\frac{A^2}{2.\,3} + \frac{A^4}{2^2.\,3^2.\,5} + \&c.\right\} \qquad (1)$$

Similar formulæ might be obtained for the logarithmic values of the cosine, tangent, &c. as follows :

$$\log.\cos A = -M\left\{\frac{A^2}{2} + \frac{A^4}{3.\,4} + \&c.\right\} \qquad (2)$$

$$\log.\tan A = \log.\,A + M.\left\{\frac{A^2}{3} + \frac{7.\,A^4}{9.\,10} + \&c.\right\} \qquad (3)$$

These three series are due to Cagnoli.—*Trig.* page 86. He would prefer, however, to find the logs. of the values in natural numbers, if our object were to form tables.

The series that have been given for sines, cosines, &c. in Chap. II. Part II. are peculiarly fitted for this purpose, as they are adapted to logarithmic computation.

Hence we have for calculating the logarithmic values of sines the following formula:

$$\log.\sin\frac{m}{n}\cdot\frac{\pi}{2}=\log.\pi+l.\frac{m}{2n}+l.\left(1-\frac{m^2}{4n^2}\right)+\&c.$$

Similar formulæ can be easily deduced for the other Trigonometrical lines.

NOTES.

Page 5. Allowing, as we do, Euler's objection to the criterion of sign that we have adopted, it may be asked why we have adopted it in preference to that of Cagnoli:—for two reasons, 1°. Not to introduce the principles of a new science; and 2°. Because the exceptions of Euler do not occur in the Elements of our Theory.

Page 7. It may be necessary also to convert English degrees into French. For this we have

$$n=\frac{n'.\,10}{9}=n'.\,(1.1111,\ \&c.)$$

or forming a small table,

English.	French.	English.	French.
1°	1°.111, &c.	10°	11°.111, &c.
1′	1′111, &c.	10′	11′.111, &c.
1″	1″.111, &c.	10″	11″.111, &c.

Page 54. The methods of finding the third side c may be more clearly stated as follows:

In the expression for $\cos c$; writing $1-2\sin^2\frac{1}{2}C$, and $2\cos^2\frac{C}{2}-1$

for $\cos C$; and $1-2\sin^2\frac{1}{2}c$, and $2\cos^2\frac{1}{2}c-1$ for $\cos c$, we obtain in all four formulæ.

$$(1)\quad \sin^2\tfrac{1}{2}c=\sin^2\tfrac{1}{2}(a-b)+\sin a.\sin b.\sin^2\tfrac{1}{2}C.$$

$$(2)\quad \cos^2\tfrac{1}{2}c=\cos^2\tfrac{1}{2}(a-b)-\sin a.\sin b.\sin^2\tfrac{1}{2}C.$$

$$(3)\quad \sin^2\tfrac{1}{2}c=\sin^2\tfrac{1}{2}(a+b)-\sin a.\sin b.\cos^2\tfrac{1}{2}C.$$

$$(4)\quad \cos^2\tfrac{1}{2}c=\cos^2\tfrac{1}{2}(a+b)+\sin a.\sin b.\cos^2\tfrac{1}{2}C.$$

Any or all of which the reader will find no difficulty in adapting to logarithmic computation by means of an auxiliary arc.

Page 68. It may be necessary, in order to complete the solution of plane and spherical triangles, to say something respecting the relations between the corresponding variations of the angles and sides of triangles; as on the knowledge of these depends in a great measure the selection of conditions most favourable to accuracy of result. The first attempt of this kind was made by Roger Cotes, in a tract entitled, '*Æstimatio errorum in mixtâ mathesi*;' and the subject has since formed a part of almost every treatise on Trigonometry.

In order to facilitate the solutions of the different cases, we shall prefix the values of the *differential co-efficients* of $\sin x$, $\cos x$, &c.

$$\frac{d.\sin x}{dx}=\cos x;\quad \frac{d.\cos x}{dx}=-\sin x;\quad \frac{d.\tan x}{dx}=\sec^2 x;$$

$$\frac{d.\sec x}{dx}=\frac{\tan x}{\cos x};\quad \frac{d.\operatorname{cosec} x}{dx}=-\frac{\cot x}{\sec x};\quad \frac{d.\cot x}{dx}=-\operatorname{cosec}^2 x.$$

Example 1.

From an error on A in a $r.\angle d$ plane $\triangle$, find the error on a.

$$a=b.\tan A,\ \therefore da=b.\frac{dA}{\cos^2 A}=\frac{2a.dA}{2.\sin A.\cos A}=\frac{2a.dA}{\sin 2A}.$$

Hence da is least when $\sin 2A$ is greatest, viz. when $A=45°$.

Example 2.

In a right angled spherical triangle A is invariable, required the ratio of the variations of c and b.

By Napier's rules $\cos A = \tan b. \cot c$. Differentiating which, we find

$$\frac{db}{dc} = \frac{\sin 2b}{\sin 2c}.$$

Example 4.

In the same let c be invariable, required the ratio of the variations of a and b.

We have $\cos c = \cos a. \cos b$. Differentiating, we have

$$\frac{da}{db} = -\tan b. \cot a.$$

Example 4.

If c vary in an oblique angled spherical triangle, required the corresponding variation in A.

Since $\cos A. \sin b. \sin c = \cos a - \cos b. \cos c$, we obtain by diff.

$$\frac{dA}{dc} = \cot A. \cot c - \cot b. \operatorname{cosec} A.$$

This shews us how, in the method of finding the time by equal altitudes, to find the error from an error on the polar distance.

Example 5.

If a vary, required the variation in A.

Differentiating the above equation, considering A and a variable, we have

$$d\,A.\ \sin A = da.\ \frac{\sin a}{\sin b.\ \sin c},$$

whence

$$\frac{d\,A}{da} = \frac{1}{\sin b.\ \sin C}.$$

This solves the astronomical question—" Given the error in altitude or zenith distance, to find the error in time."

It would be easy to enlarge extensively on this subject, but it would be rather inconsistent with an elementary treatise to do so. We trust the few examples that we have given sufficiently explain this part of our subject.

Page 79. From what has been stated in this Chapter, the reader will find no difficulty in summing the following series:

$$\cos A \pm 2^m.\ \cos 2\,A + 3^m.\ \cos 3\,A \pm \&c.\ \textit{ad inf.}$$

$$\sin A \pm 2^m.\ \sin 2\,A + 3^m.\ \sin 2\,A \pm \&c.\ \textit{ad inf.}$$

m being a positive integer.

$$1 \pm 2^m + 3^m \pm 4^m\ \&c.\ \textit{ad inf.}$$

$$\cos\left(\frac{\pi}{4} \pm x\right) + \cos\left(\frac{\pi}{4} \pm 2\,x\right) \ldots\ldots + \cos\left(\frac{\pi}{4} \pm nx\right)$$

$$\cos A + \tfrac{1}{2}.\ \cos 2\,A + \tfrac{1}{3}.\ \cos 3\,A\ \textit{ad inf.}$$

$$\sin A + \tfrac{1}{2}.\ \sin 2\,A + \tfrac{1}{3}.\ \sin 3\,A\ \textit{ad inf.}$$

Page 83. This elegant Theorem which we have seen to comprise all the formulæ of verification that have been given for finding the numerical values of Trigonometrical lines, admits of a Geometrical demonstration, as follows:

Let the several subtenses of one, two, three, &c. arcs be called in their order s', s'', s'''............ s^n; then we have by Cor. (4), Prop. 16. Book VI. *Elr.* the following Theorems:

$$c'' s' = s''. c' + s'. c^n \qquad c' s' = c'. s'$$

$$c^{iv}. s' = s^{iv}. c' + s'''. c^n \qquad c'''. s' = c^n s'' + c'. s''$$

$$c^{vi}. s' = s^{vi}. c' + s^{v}. c^n \qquad c^{v}. s' = c^n. s^{iv} + c'. s^{v}$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$c^{n-1}. s' = s^{n-1}. c' + s^{n-2}. c^n \qquad c^{n-2}. s' = c^n. s^{n-3} + c'. s^{n-2}$$

Attending to the fact that $s^{n-p} = s^p$, we can instantly perceive that the right hand members of these sets of equations are equal, so $\therefore$ must the left, hence we have

$$c' + c''' + c^{v} \ldots\ldots c^n = c'' + c^{iv} + c^{vi} \ldots\ldots c^{n-1}.$$

Page 101, Art. 1. By the theory of quadratic equations if $u = x^2 + px + q$, and if x', x'', be the values of x that fulfill the equation $x^2 + px + q = 0$, then

$$u = (x - x'). (x - x'');$$

hence if $u = \cos^2 A + \sin^2 A$, from the equation $\cos^2 A + \sin^2 A = 0$, we have

$$\cos A = \pm \sqrt{-1}. \sin A, \ \therefore u = (\cos A + \sqrt{-1}. \sin A).$$
$$(\cos A - \sqrt{-1}. \sin A).$$

Page 101, Art. 2. Hence we have the following beautiful and general Theorem:

$$(\cos A \pm \sqrt{-1}. \sin A). (\cos B \pm \sqrt{-1}. \sin B).$$
$$(\cos C \pm \sqrt{-1}. \sin C). \&c.$$

$$= \cos (A + B + C + \&c.) \pm \sqrt{-1}. \sin (A + B + C + \&c.)$$

Page 104, Line 12. This line may be better expressed as follows:

" Using $\frac{A'}{n}$ for A, and taking the series (1) and (2) in their limits, we have, &c."

Page 105, Art. 7. The attentive reader may find reason to think not only this, but every attempt that has been made to extend De Moivre's Theorem to surd and imaginary values, fallacious at bottom. It would be necessary for writers on the calculus to settle whether any meaning can be attached to the differentiation of a sine or cosine of a surd or imaginary arc.

Page 107. Series (7) is the only one, the obtaining of which is easier by the calculus than by the methods we have given. By the integral calculus we have it as follows:

$$dA = \frac{d. \sin A}{\cos A} = d. \sin A\,(1 - \sin^2 A)^{-\frac{1}{2}}$$

$$= d. \sin A. \left\{1 + \tfrac{1}{2}. \sin^2 A + \tfrac{1}{2}. \tfrac{3}{4}. \sin^4 A + \&c.\right\}$$

Integrating, we obtain

$$A = C + \left\{\sin A + \tfrac{1}{2}. \frac{\sin^3 A}{3} + \tfrac{1}{2}. \tfrac{3}{4}. \frac{\sin^5 A}{5} + \&c.\right\}.$$

The constant $= 0$ since the arc and sine begin together.

Page 130, Line 7. The author suspected some latent fallacy in the method here proposed for finding $\cos^n x$, by adding the two developements $\left(x + \frac{1}{x}\right)^n$ and $\left(\frac{1}{x} + x\right)^n$. This was the cause of his preferring the method of Lagrange. That the suspicion was justly entertained will appear from the exposure of the fallacy by the ingenious M. Poisson, which the reader will find in Lacroix *Calc. Diff.* tom. III. page 605, or in the 2d vol. of the *Correspondance sur l'Ecole Polytechnique.*

Page 132, Art. 31. Let us seek the developement of $\sin (x+h)$, in a series ascending by the powers of h, by the Theorem of Taylor.

Let $u' = \sin (x+h)$, then

$$\frac{du}{dx} = \cos x; \ \frac{d^2u}{dx^2} = -\sin x; \frac{d^3u}{dx^3} = -\cos x, \ \&c.$$

$\therefore$ we have

$$\sin(x+h) = \sin x + \cos x.\frac{h}{1} - \sin x.\frac{h^2}{1.2} - \cos x\frac{h^3}{1.2.3} + \&c.$$

$$= \left\{ \begin{array}{l} \sin x.\left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4.} - \&c.\right) \\ + \cos x.\left(h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \&c.\right) \end{array} \right\}$$

$$= \sin x.\cos h + \cos x.\sin h.$$

This is another sophistical foundation for a system of Trigonometry.

Page 156. The method here given of determining the angle of the chords has been adopted at the suggestion of a friend. It is to be found in a note to an American edition of Hutton's Mathematics.

Page 189, Art. 27. The method of converting a series into a continued fraction, the reader will find in Garnier *Analyse Algebrique*, page 26.

THE END.

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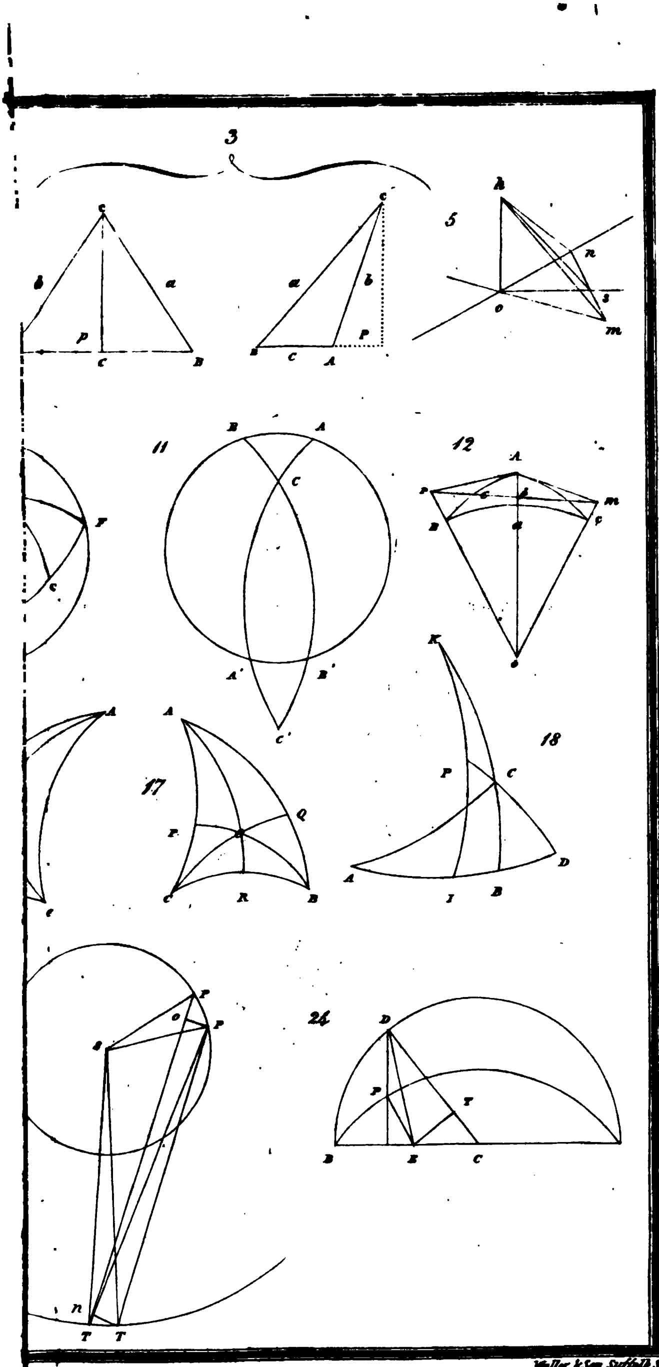
ERRATA.

Page	Line	For	Read
8	13	*mo*	*so*
14	2	(f)	(q)
15	formula (*)	$\frac{1}{2}B$	B
28	11	f	(q)
32	36	*CSD*	*CSB*
34	2	willl	will
61	11	sphære	sphere
61	15	oo	oo'
63	19	sin c	sin a
64	8	β	b
84	8	(Fig. 16)	(Fig. 19)
96	4 & 12	Herchell	Herschell
96	14	system	sine
98	14	cos 3 A	cos 3 nA
98	18	suries	series
102	17	$\pm$	$\mp$
112	13	She	The
113	14	(Fig. 17)	(Fig. 20)
149	15	$\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}$	$\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$
149	16	$+$	— between the two radicals
160	14	*BEB*	*BEP*



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